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$N = 2$ supergravity in five dimensions revisited

Eric Bergshoeff[1], Sorin Cucu[2], Tim de Wit[1],

Jos Gheerardyn[2], Stefan Vandoren[3]

and Antoine Van Proeyen[2]

[1] Center for Theoretical Physics, University of Groningen,
Nijenborgh 4, 9747 AG Groningen, The Netherlands.
{e.a.bergshoeff, t.c.de.wit}@phys.rug.nl

[2] Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven,
Celestijnenlaan 200D B-3001 Leuven, Belgium.
sorin.cucu@ua.ac.be, {jos.gheerardyn, antoine.vanproeyen}@fys.kuleuven.ac.be

[3] Institute for Theoretical Physics, Utrecht University,
Leuvenlaan 4, 3508 TA Utrecht, The Netherlands.
s.vandoren@phys.uu.nl

Abstract

We construct matter-coupled $N = 2$ supergravity in five dimensions, using the superconformal approach. For the matter sector we take an arbitrary number of vector, tensor and hypermultiplets. By allowing off-diagonal vector-tensor couplings we find more general results than currently known in the literature. Our results provide the appropriate starting point for a systematic search for BPS solutions, and for applications of M-theory compactifications on Calabi-Yau manifolds with fluxes.

Contents

1 Introduction

Matter-coupled supergravities in five dimensions have attracted renewed attention [1,2] due to the important role they play in the Randall-Sundrum (RS) braneworld scenario [3,4], in M-theory compactifications on Calabi-Yau manifolds [5] with applications to flop transitions and cosmology [6,7], and in the $\mathrm{AdS}_6/\mathrm{CFT}_5$ [8,9] and $\mathrm{AdS}_5/\mathrm{CFT}_4$ [10] correspondences. Generically, one is interested in studying BPS solutions, such as domain walls or black holes, or in vacua obtained from flux-compactifications, perhaps even in time-dependent solutions with a non-vanishing cosmological constant. In many of these applications, a crucial role is played by the properties of the scalar potential that appears after coupling matter multiplets to $N = 2$ supergravity. For example, the possibility for finding a supersymmetric RS scenario depends on the existence of a domain-wall solution (which requires a scalar potential) containing a warp factor with the correct asymptotic behaviour such that gravity is suppressed in the transverse direction. It is proved in [11,12] that one cannot restrict oneself to vector multiplets, but that hypermultiplets are needed. Some interesting solutions have already been found in [13,14].

In order to make a more systematic search for the variety of BPS solutions such as domain walls or black holes that appear in five dimensions, we need to know the most general form of matter-coupled supergravity. It is the purpose of this work to construct these matter couplings. Actually, the present work is the third and last in a series of papers where we apply the superconformal programme to derive these matter couplings (see also [15–17]). The first paper dealt with the construction of the $N = 2$ conformal supergravity multiplet [18], while in the second paper we presented the superconformal matter couplings [19]. In the present work we will perform the last step in the superconformal programme, i.e. perform the gauge-fixing and obtain the matter-coupled $N = 2$, $D = 5$ Poincaré supergravity theory.

Quite some work on $N = 2$ matter-coupled supergravities in five dimensions has already been done. Since we claim to give more general results than currently known in the literature, we first summarize what has been done. The pure supergravity sector was constructed in [20]. The coupling to vector multiplets was given in [21,22]. More recently, the addition of tensor multiplets was considered in [1,23]. There it was stated that certain couplings between vector and tensor multiplets were impossible to supersymmetrize in a gauge-invariant way (except possibly in very special cases). In this work we will show that such couplings are possible and can be supersymmetrized thereby generalizing the results of [1,23]. Finally, vector, tensor and hypermultiplets were treated together in [2].

The superconformal programme, apart from leading us to general matter couplings, has another bonus. It is well known that the scalars of the matter sector can be viewed as the coordinates on a manifold. It turns out that there are interesting relations between the geometries of these scalar manifolds before and after gauge-fixing the superconformal symmetries. In [24,25], followed by [19], this was demonstrated for hypermultiplet scalars. The geometries before gauge-fixing are hypercomplex or hyper-Kähler dependent on whether there exists an action or not (see [19] for more details). After gauge-fixing, the relevant geometries are quaternionic and quaternionic-Kähler, respectively. Since the relations between these geometries are interesting in themselves, and can be studied independently of the

present context, we are preparing a companion paper where we present the details about all the geometries involved and their relations [26]. Sometimes we will refer in this paper to [26] for more details on the geometry.

The conformal programme has already been discussed extensively at several occasions (see e.g. [27, 28]) including our previous paper [19]. We refer the reader to these reviews for more details. We will attempt to give a flavour of the conformal approach by presenting a toy model. More explicitly, consider a scalar-gravity model in four dimensions. We start with a conformally invariant action for a scalar field ϕ

$$\mathcal{L} = \sqrt{|g|} \left[\tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{12}R\phi^2 \right], \tag{1.1}$$

which is invariant under the following local dilatations (with parameter $\Lambda_D(x)$)

$$\delta\phi = \Lambda_D\phi, \qquad \delta g_{\mu\nu} = -2\Lambda_D g_{\mu\nu}. \tag{1.2}$$

This dilatation symmetry can be gauge fixed by choosing the gauge

$$\phi = \frac{\sqrt{6}}{\kappa}. \tag{1.3}$$

This leads to the Poincaré action

$$\mathcal{L} = \frac{1}{2\kappa^2}\sqrt{|g|}R. \tag{1.4}$$

Therefore, the actions (1.1) and (1.4) are *gauge equivalent*. Alternatively, we could have chosen new coordinates ($g'_{\mu\nu} = g_{\mu\nu}\kappa^2\phi^2$), such that the resulting action is manifestly invariant under the dilatation symmetry. Although ϕ still transforms under dilatations, the field does not appear in the action anymore. The scalar ϕ has no physical degrees of freedom, and is called a 'compensating scalar'. Note that the scalar kinetic term has the wrong sign; this is a generic feature of compensating scalars which we will also encounter in the more complicated case of conformal supergravity.

The same mechanism will be used in this paper to obtain five-dimensional matter-coupled Poincaré supergravity. To this end, the Poincaré algebra is first extended to the local super-conformal algebra $F^2(4)$. In our first paper [18] we constructed the minimal representation of the superconformal algebra containing the graviton, called the standard Weyl multiplet. This multiplet plays the role of $g_{\mu\nu}$ in the above toy model. It turns out that in the case of $N = 2$, $D = 5$ supergravity we need one hypermultiplet and one vector multiplet as compensators. They play the role of the compensating scalar ϕ in the above toy model. We thus have

$$\begin{aligned} g_{\mu\nu} &\rightarrow \text{ standard Weyl multiplet,} \\ \phi &\rightarrow \text{ 1 hyper- + 1 vector-multiplet.} \end{aligned} \tag{1.5}$$

On top of this we add an arbitrary number of n_V vector, n_T tensor and n_H hypermultiplets. We thus end up with $(n_V + 1)$ vector, n_T tensor and $(n_H + 1)$ hypermultiplets. As explained in [1, 19, 23], the tensor generically is part of a vector-tensor multiplet which is a hybrid form of a vector and a tensor multiplet. The label of the vector-tensor multiplet has $n_V + 1$

vector-multiplet and n_T tensor-multiplet directions. Our starting point is therefore a number of vector, tensor and hypermultiplets coupled to conformal supergravity:

$$\mathcal{L}_{\text{Total}} = \mathcal{L}_{\text{Vector}-\text{Tensor}} + \mathcal{L}_{\text{Hyper}}. \tag{1.6}$$

These conformal couplings, which are the analogue of (1.1) in the toy model, have been constructed in our second paper [19], and are repeated in appendix B to keep the presentation self-contained. In the main part of this paper we will discuss the final step, i.e. the gauge-fixing, where we get rid of all the superconformal symmetries that are not part of the super-Poincaré algebra. This is the analogue of (1.3), leading to a result similar to (1.4). The main goal of this paper is to derive the analogue of (1.4). In the present case we end up with n_V vector, n_T tensor and n_H hypermultiplets coupled to $N = 2$, $D = 5$ Poincaré supergravity. The final answer is given in (5.7). For the actual purpose of searching for supersymmetric BPS solutions we only need the bosonic terms in the action. These have been collected in (7.1).

The organization of this paper is as follows. In section 2, we give the field content of the standard Weyl multiplet and the different matter multiplets: vector, tensor and hyper. The corresponding supersymmetry and superconformal transformation rules, together with the invariant actions are given in appendix B. Next, some details about the conformal geometry for hypermultiplets, i.e. the geometry before gauge-fixing, and its relation to the geometry after gauge-fixing, are presented in section 3. The gauge-fixing procedure is discussed in section 4, accompanied by some well-known properties of very special geometry, listed in appendix C. The resulting action after gauge-fixing is given in section 5. In section 6, we compare our results with the existing literature. Finally, in section 7 we collect those terms in the action and supersymmetry rules that will be relevant to our search for BPS solutions.

We use the same notation as in our previous two papers except that the sign of the spacetime Ricci tensor and Ricci scalar has changed. For the convenience of the reader we have collected the definitions of all curvatures in appendix A. Further details of the notation can be found in the appendix of our first paper [18].

2   Multiplets

The fields of the standard Weyl multiplet and their properties are listed in table 1. The full details of this multiplet are given in [18]. We use the following notation for indices: $\mu(a)$ are curved (flat) world indices with $\mu, a = 0, 1, 2, 3, 4$ and $i = 1, 2$ is an $SU(2)$ index. All fermions are symplectic Majorana spinors. In [19] we constructed vector-tensor multiplets and hypermultiplets in the background of this Weyl multiplet. Vector-tensor multiplets are a hybrid form of vector and tensor multiplets. The field content and further properties of these multiplets are given in table 2. Here we have introduced the following indices: $I = 0, \cdots, n_V$ labels the adjoint representation of some gauge group G, and $M = n_V + 1, \cdots, n_V + n_T$ labels some representation of G, possibly reducible, under which the tensors transform (see below for more details). Finally, $X = 1, \cdots, 4(n_H + 1)$ and (i, A) with $i = 1, 2$ and $A = 1, \cdots, 2(n_H + 1)$ are, respectively, the curved and flat indices of the

Field	#	Gauge	$SU(2)$	w
		Elementary gauge fields		
$e_\mu{}^a$	9	P^a	1	-1
b_μ	0	D	1	0
$V_\mu^{(ij)}$	12	$SU(2)$	3	0
ψ_μ^i	24	Q^i	2	$-\frac{1}{2}$
		Dependent gauge fields		
$\omega_\mu{}^{ab}$	$-$	$M^{[ab]}$	1	0
$f_\mu{}^a$	$-$	K^a	1	1
ϕ_μ^i	$-$	S^i	2	$\frac{1}{2}$
		Matter fields		
$T_{[ab]}$	10		1	1
D	1		1	2
χ^i	8		2	$\frac{3}{2}$

Table 1: *Fields of the 32 + 32 standard Weyl multiplet. The symbol # indicates the off-shell degrees of freedom. The first block contains the (bosonic and fermionic) gauge fields of the superconformal algebra. The fields in the middle block are dependent gauge fields. The extra matter fields that appear in the standard Weyl multiplet are displayed in the lower block. Note that we have suppressed the spinor index of both the Q and S generators, of the corresponding gauge fields and of the matter field χ^i.*

hypermultiplet scalar manifold[1]. Note that we have introduced $n_V + 1$ vector multiplets and $n_H + 1$ hypermultiplets to indicate that one of the vector multiplets and one of the hypermultiplets serve as compensating multiplets. When combining vectors and tensors into a vector-tensor multiplet we will sometimes write $\tilde{I} = (I, M)$.

We first consider the vector-tensor multiplet. A remarkable feature of the vector-tensor multiplet is that the vector-part is off-shell whereas the tensor-part is on-shell. The gauge transformations of the vector-tensor multiplet are specified by matrices $(t_I)_{\tilde{J}}{}^{\widetilde{K}}$ that satisfy the commutation relations (with structure constants $f_{IJ}{}^K$)

$$[t_I, t_J] = -f_{IJ}{}^K t_K. \tag{2.1}$$

These gauge transformations (with parameters Λ^I) are given by

$$\delta_G(\Lambda^J) A_\mu^I = \partial_\mu \Lambda^I + g A_\mu^J f_{JK}{}^I \Lambda^K, \qquad \delta_G(\Lambda^J) X^{\tilde{I}} = -g \Lambda^J (t_J)_{\widetilde{K}}{}^{\tilde{I}} X^{\widetilde{K}}, \tag{2.2}$$

---
[1]For comparison with our previous paper [19]: $n = n_V + 1$, $\qquad m = n_T$, $\qquad r = n_H + 1$.

Field	$SU(2)$	w	# d.o.f.
	off-shell vector multiplet		
A_μ^I	1	0	$4(n_V + 1)$
Y^{ijI}	3	2	$3(n_V + 1)$
σ^I	1	1	$1(n_V + 1)$
ψ^{iI}	2	3/2	$8(n_V + 1)$
	on-shell tensor multiplet		
$B_{\mu\nu}^M$	1	0	$3n_T$
Y^{ijM}	3	2	0
σ^M	1	1	$1n_T$
ψ^{iM}	2	3/2	$4n_T$
	on-shell hypermultiplet		
q^X	2	3/2	$4(n_H + 1)$
ζ^A	1	2	$4(n_H + 1)$

Table 2: *The relevant $D = 5$ superconformal matter multiplets. We introduce $(n_V + 1)$ off-shell vector multiplets, n_T on-shell tensor multiplets and $(n_H + 1)$ on-shell hypermultiplets. Indicated are their $SU(2)$ representations, Weyl weights w and the number of off-shell/on-shell degrees of freedom. Each of the multiplets describes $4 + 4$ on-shell degrees of freedom.*

where $X^{\widetilde{I}}$ is a general matter field and g is the coupling constant of the group G. Closure of the supersymmetry algebra requires $(t_J)_M{}^I = 0$, so the generic form of the matrices t_I is given by

$$(t_I)_{\widetilde{J}}{}^{\widetilde{K}} = \begin{pmatrix} f_{IJ}{}^K & (t_I)_J{}^N \\ 0 & (t_I)_M{}^N \end{pmatrix}, \qquad \begin{cases} I, J, K & = & 0, \ldots, n_V \\ M, N & = & n_V + 1, \ldots, n_V + n_T. \end{cases} \tag{2.3}$$

Sometimes we extend the index I in $(t_I)_{\widetilde{J}}{}^{\widetilde{K}}$ to $\widetilde{I}$ with the understanding that

$$(t_M)_{\widetilde{J}}{}^{\widetilde{K}} = 0. \tag{2.4}$$

If $n_T \neq 0$ then the representation $(t_I)_{\widetilde{J}}{}^{\widetilde{K}}$ is reducible. In our second paper [19] we showed that this representation can be more general than assumed so far in treatments of vector-tensor couplings. In particular, the off-diagonal matrix elements $(t_I)_J{}^N$ lead to new matter couplings, and the requirement that n_T is even will only appear when we demand the existence of an action or if we require the absence of tachyonic modes. The supersymmetry rules of the vector-tensor multiplet can be found in appendix B.

In the absence of an action, the vector-tensor multiplet is characterized by the matrices t_I. In order to write down a superconformal action we need to introduce two more symbols:

a fully symmetric tensor $C_{\widetilde{I}\widetilde{J}\widetilde{K}}$ and an antisymmetric and invertible tensor Ω_{MN}. They are related by

$$
\begin{aligned}
C_{M\widetilde{J}\widetilde{K}} &= t_{(\widetilde{J}\widetilde{K})}{}^P \Omega_{PM}, \\
t_{I[M}{}^P \Omega_{N]P} &= 0, \\
t_{I(\widetilde{J}}{}^{\widetilde{M}} C_{\widetilde{K}\widetilde{L})\widetilde{M}} &= 0,
\end{aligned}
\tag{2.5}
$$

see section 3.2 of [19]. The corresponding action is given in appendix B.

We now turn to the hypermultiplet. In the absence of an action, the superconformal tensor calculus performed in [19] resulted in the construction of a hypercomplex manifold spanned by the $(4n_H + 4)$ hyperscalars q^X. This manifold includes the four scalars of the compensating hypermultiplet. The geometrical properties of the hypercomplex manifold are determined by a collection of vielbeins f_X^{iA}, satisfying the following constraints:

$$
\begin{aligned}
f_Y^{iA} f_{iA}^X &= \delta_Y^X, \qquad f_X^{iA} f_{jB}^X = \delta_j^i \delta_B^A, \\
\mathfrak{D}_Y f_{iB}^X &\equiv \partial_Y f_{iB}^X - \omega_{YB}{}^A f_{iA}^X + \Gamma_{ZY}{}^X f_{iB}^Z = 0,
\end{aligned}
\tag{2.6}
$$

where $\Gamma_{ZY}{}^X$ can be interpreted as the affine connection on the manifold, and $\omega_{YB}{}^A$ as the $G\ell(n_H + 1, \mathbb{H})$ connection. The last constraint shows that the vielbeins are covariantly constant with respect to the connections Γ and ω. These connections do not represent independent functions, see below. On this manifold we introduce a triplet of complex structures, defined as

$$
\vec{J}_X{}^Y \equiv -\mathrm{i} f_X^{iA} \vec{\sigma}_i{}^j f_{jA}^Y.
\tag{2.7}
$$

Using (2.6), they are covariantly constant and satisfy the quaternion algebra, which is that for any vectors $\vec{A}$ and $\vec{B}$,

$$
\vec{A} \cdot \vec{J}_X{}^Z \, \vec{B} \cdot \vec{J}_Z{}^Y = -\delta_X^Y \vec{A} \cdot \vec{B} + (\vec{A} \times \vec{B}) \cdot \vec{J}_X{}^Y,
\tag{2.8}
$$

At some places we will use a doublet notation instead of the triplet (or vector) notation:

$$
J_X{}^Y{}_i{}^j \equiv \mathrm{i} \vec{J}_X{}^Y \cdot \vec{\sigma}_i{}^j = 2 f_X^{jA} f_{iA}^Y - \delta_i^j \delta_X^Y.
\tag{2.9}
$$

The same transition between doublet and triplet notation is also used for other quantities in the adjoint representation of $SU(2)$.

Note that the complex structure is obtained from the vielbeins. Its covariant constancy is sufficient to determine the affine connection. The latter is then called the Obata connection (similar to the definition of a Levi-Civita connection determined from the covariant constancy of a metric). Further, once this Obata connection is known, the covariant constancy of the vielbeins as in the last line of (2.6) determines the $G\ell(n_H + 1, \mathbb{H})$ connection $\omega_{XA}{}^B$. In this way all quantities can be derived from the vielbeins.

In order to build local supergravity theories, we require the hyperscalar manifolds that we will use to be 'conformal' in the sense that they contain a homothetic Killing vector k^X, describing the dilatations of q^X [24, 29]

$$
\mathfrak{D}_Y k^X = \tfrac{3}{2} \delta_Y^X.
\tag{2.10}
$$

Using the complex structures this defines $SU(2)$ symmetry generators $\vec{k}^X$:

$$\vec{k}^X \equiv \tfrac{1}{3} k^Y \vec{J}_Y{}^X. \tag{2.11}$$

Both k^X and $\vec{k}^X$ are discussed in section 2.3.2 of [19].

In section 2.3.3 of [19], we also considered the action of the symmetry group gauged by the vector multiplets on the hypercomplex manifold. Their generators are therefore labelled by the index I. They are parametrized by triholomorphic (i.e. leaving the complex structures invariant) Killing[2] vectors k_I^X:

$$
\begin{aligned}
\delta_G q^X &= -g \Lambda_G^I k_I^X(q), \\
\delta_G \zeta^A &= -g \Lambda_G^I t_{IB}{}^A(q) \zeta^B - \zeta^B \omega_{XB}{}^A \delta_G q^X, \\
t_{IB}{}^A &\equiv \tfrac{1}{2} f_{iB}^Y \mathfrak{D}_Y k_I^X f_X^{iA}, \qquad f_A^{Y(i} f_X^{j)B} \mathfrak{D}_Y k_I^X = 0.
\end{aligned} \tag{2.12}
$$

The supersymmetry rules of the hypermultiplet are given in appendix B.

So far, we have not assumed the existence of an action. In order to write down an action we need to introduce a covariantly constant antisymmetric invertible tensor C_{AB} which will be used to raise and lower the A indices. We can now construct the metric on the scalar manifold as

$$g_{XY} = f_X^{iA} C_{AB} \epsilon_{ij} f_Y^{jB}, \tag{2.13}$$

and the hypercomplex manifold becomes a hyper-Kähler manifold. The Obata connection then coincides with the Levi-Civita connection. The action also contains four-Fermi terms, proportional to a tensor $W_{ABC}{}^D$. The latter is defined in terms of the Riemann tensor and conversely completely determines the Riemann tensor:

$$W_{ABC}{}^D = \tfrac{1}{2} f_A^{iX} f_{iB}^Y f_{jC}^Z f_W^{Dj} R_{XYZ}{}^W, \qquad R_{XYW}{}^Z = -\tfrac{1}{2} f_X^{Ai} \varepsilon_{ij} f_Y^{jB} f_W^{kC} f_{kD}^Z W_{ABC}{}^D, \tag{2.14}$$

see appendix B of [19]. The metric breaks the holonomy group from $G\ell(n_H + 1, \mathbb{H})$ to $USp(2, 2n_H)$, where we chose the metric to have signature $(----++\ldots+)$, as it is required for a physical theory with positive kinetic terms. The $-$ signs correspond to the scalars of the compensating multiplet. This is to be compared with the negative kinetic energy of the compensator in the toy model discussed in the introduction. From the integrability condition that follows from the covariant constancy of C_{AB}, and using a basis with constant C_{AB}, we can determine that the connection ω_X is symmetric, and thus

$$\omega_{XAB} \equiv \omega_{XA}{}^C C_{CB} = \omega_{XBA}, \tag{2.15}$$

is now the connection for $USp(2, 2n_H)$.

Finally, when an action exists, the symmetries should respect the metric, i.e. the Killing equation

$$\mathfrak{D}_{(X} k_{Y)I} = 0, \tag{2.16}$$

---

[2] The word 'Killing vector' is in fact only appropriate when they respect a metric, see (2.16), which is the case that we will mostly consider in this paper.

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should be satisfied. Then, moment maps $\vec{P}_I$ can be defined, see section 3.3.2 of [19],

$$\partial_X \vec{P}_I = -\tfrac{1}{2}\vec{J}_{XY}k_I^Y,\tag{2.17}$$

which by the conformal symmetry are determined to be

$$\vec{P}_I = -\tfrac{1}{6}k^X \vec{J}_X{}^Y k_I^Z g_{YZ}.\tag{2.18}$$

They appear in the scalar potential of the action, which is given in appendix B. This discussion of isometries and moment maps on such hyper-Kähler spaces applies as well in 6 and 4 spacetime dimensions, and appeared in the 4-dimensional theories in [30].

3   Geometry

In preparation for the later developments, but also as an introduction to the geometrical aspects of the gauge-fixing procedure discussed hereafter, we recall in this section some ideas concerning the connection between conformal hypercomplex (hyper-Kähler) and quaternionic(-Kähler) manifolds. The details of the explicit map between the corresponding geometries are presented in the companion paper [26]. *For the rest of this paper we will always assume the presence of a metric. Therefore, we will only deal with the hyper-Kähler and quaternionic-Kähler geometries.* The geometry and relation between these spaces were also analysed in [25], and in the mathematics literature [31]. Our analysis here involves a different choice of coordinates and gauge-fixing procedure than in [25], and will turn out to be more convenient for our purposes. The case without a metric, i.e. the map between the hypercomplex and quaternionic geometries, is discussed in [26].

In the superconformal tensor calculus there is one vector and one hypermultiplet that play a special role as a compensating multiplets. We choose 3 scalars of the hypermultiplet to gauge-fix the three $SU(2)$ gauge transformations and 1 to gauge-fix the dilatation D. This means that 4 compensating scalars will be removed from the hyper-Kähler manifold. In view of this, it is convenient to split these coordinates off in a manifest way by making a specific coordinate choice on the hyper-Kähler manifold. Full proofs and the exact mapping between arbitrary hypercomplex and quaternionic spaces in this way will be published in [26]. Here we will summarize the relevant results. From now on we will use the hat-notation for objects that are defined on the 'higher-dimensional' hyper-Kähler manifold. For instance, the coordinates of the hyper-Kähler manifold are denoted by $q^{\hat{X}}$ ($\hat{X} = 1, \cdots, 4n_H + 4$).

The three isometries generated by the three $SU(2)$ Killing vectors $\vec{\hat{k}}^{\hat{X}}$ are gauged using the vectors of the Weyl multiplet. Using Frobenius' theorem it is shown in [26] that the three-dimensional subspace spanned by the directions of the three $SU(2)$ transformations can be parametrized by coordinates z^α (with $\alpha = 1, 2, 3$). Furthermore, using the homothetic Killing equation, one coordinate z^0 can be associated with the dilatation transformation. The remaining directions are indicated by q^X ($X = 1, \ldots, 4n_H$). Thus, we split the coordinates on the hyper-Kähler manifold as

$$\{q^{\hat{X}}\} = \{z^0,\, z^\alpha,\, q^X\}.\tag{3.1}$$

Throughout this paper we will work in this coordinate basis. In this basis, the dilatation and $SU(2)$ transformations, see (2.10) and (2.11), take on the following form [26]:

$$\hat{k}^{\hat{X}}(z,q) = \{3z^0, 0, 0\}, \qquad \vec{\hat{k}}^{\hat{X}}(z,q) = \{0, \vec{k}^\alpha(z,q), 0\}. \tag{3.2}$$

Thus in the bosonic sector only z^0 transforms under dilatations and only z^α transforms under the $SU(2)$ in the superconformal group.

Similarly, we can split the tangent space index $\hat{A}$ as $\{\hat{A}\} = \{i, A\}$ ($i = 1, 2$; $A = 1, \ldots, 2n_H$), where i is an $SU(2)$ index, implying

$$\{\zeta^{\hat{A}}\} = \{\zeta^i, \zeta^A\}. \tag{3.3}$$

This basis is chosen such that in the fermionic sector only ζ^i transforms under S-supersymmetry.

Further analysis of the dilatations and SU(2) isometries shows that, in these coordinates, the metric takes the form of a cone over a tri-Sasakian manifold, see e.g. [32],

$$
\begin{aligned}
\mathrm{d}\hat{s}^2 &\equiv \hat{g}_{\hat{X}\hat{Y}} \mathrm{d}q^{\hat{X}} \mathrm{d}q^{\hat{Y}}, \\
&= -\frac{(\mathrm{d}z^0)^2}{z^0} + z^0 \left\{ h_{XY}(q) \, \mathrm{d}q^X \, \mathrm{d}q^Y \right. \\
&\quad \left. - g_{\alpha\beta}(z,q) \left[\mathrm{d}z^\alpha + A_X^\alpha(z,q) \, \mathrm{d}q^X \right] \left[\mathrm{d}z^\beta + A_Y^\beta(z,q) \, \mathrm{d}q^Y \right] \right\},
\end{aligned}
\tag{3.4}
$$

where we have chosen the signs and factors for later convenience. We have defined

$$A_X^\alpha(z,q) \equiv \hat{f}_{ij}^\alpha \hat{f}_X^{ij} = -\hat{f}_{iA}^\alpha \hat{f}_X^{iA}, \qquad \hat{g}_{\alpha\beta}(z,q) \equiv z^0 g_{\alpha\beta}. \tag{3.5}$$

The latter is an (invertible) metric in the z^α-space, used to raise and lower α, β indices. It turns out that, for each value of z^0,

$$g_{XY}(z,q) \equiv z^0 h_{XY}(q) = \hat{g}_{XY} + \hat{g}_{\alpha\beta} A_X^\alpha A_Y^\beta \tag{3.6}$$

defines a quaternionic-Kähler metric on the base-space spanned by the coordinates q^X.

The $\vec{k}^\alpha$ generate an $SU(2)$ algebra, which is the statement that

$$\vec{k}^\gamma \times \partial_\gamma \vec{k}^\alpha = \vec{k}^\alpha. \tag{3.7}$$

$\vec{k}_\alpha \equiv \hat{g}_{\alpha\beta} \vec{k}^\beta$ are proportional to the inverse of $\vec{k}^\alpha$ as 3×3 matrices:

$$\vec{k}^\alpha \cdot \vec{k}_\beta = -z^0 \delta_\beta^\alpha. \tag{3.8}$$

The dependence of these $SU(2)$ Killing vectors and of A_X^α on z^0 and z^α and q^X is further restricted by

$$
\partial_0 \vec{k}^\alpha = 0, \qquad \partial_0 \left(\frac{\vec{k}_\alpha}{z^0} \right) = 0, \qquad \partial_0 \vec{A}_X = 0, \qquad \vec{A}_X \equiv \frac{1}{z^0} A_X^\alpha \vec{k}_\alpha,
$$
$$
\left(z^0 \partial_\alpha - \vec{k}_\alpha \times \right) \vec{A}_X = \partial_X \vec{k}_\alpha, \qquad \partial_{[X} \vec{A}_{Y]} - \tfrac{1}{2} \vec{A}_X \times \vec{A}_Y = \tfrac{1}{2} \vec{J}_{[X}{}^Z h_{Y]Z}. \tag{3.9}
$$

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In this basis[3] we find the following expressions for the vielbeins $\hat{f}_{\hat{X}}^{i\hat{A}}$:

$$\hat{f}_0^{ij} = \mathrm{i}\varepsilon^{ij}\sqrt{\tfrac{1}{2z^0}}, \qquad \hat{f}_\alpha^{ij} = \sqrt{\tfrac{1}{2z^0}}\vec{k}_\alpha \cdot \vec{\sigma}^{ij}, \qquad \hat{f}_X^{ij} = \sqrt{\tfrac{z^0}{2}}\vec{A}_X \cdot \vec{\sigma}^{ij},$$
$$\hat{f}_0^{iA} = 0, \qquad\qquad \hat{f}_\alpha^{iA} = 0, \qquad\qquad\quad \hat{f}_X^{iA} = f_X^{iA}, \tag{3.10}$$

where $f_X^{iA}(q)$ are the quaternionic-Kähler vielbeins. The inverse vielbeins $\hat{f}_{i\hat{A}}^{\hat{X}}$ are given by

$$\hat{f}_{ij}^0 = -\mathrm{i}\varepsilon_{ij}\sqrt{\tfrac{1}{2}z^0}, \qquad \hat{f}_{ij}^\alpha = \sqrt{\tfrac{1}{2z^0}}\vec{k}^\alpha \cdot \vec{\sigma}_{ij}, \qquad \hat{f}_{ij}^X = 0,$$
$$\hat{f}_{iA}^0 = 0, \qquad\qquad\quad \hat{f}_{iA}^\alpha = -f_{iA}^X A_X^\alpha, \qquad \hat{f}_{iA}^X = f_{iA}^X, \tag{3.11}$$

where $f_{iA}^X(q)$ are the inverse quaternionic-Kähler vielbeins. Note that we have chosen our coordinates such that $\hat{f}_{ij}^X = 0$. This means that from the supersymmetry rule $\delta q^{\hat{X}} = -\mathrm{i}\bar{\epsilon}^i\zeta^{\hat{A}}f_{i\hat{A}}^{\hat{X}}$ it follows that q^X only transforms to ζ^A and not to ζ^i.

For the complex structures we have

$$\widehat{\vec{J}}_0{}^0 = 0, \qquad \widehat{\vec{J}}_\alpha{}^0 = \vec{k}_\alpha, \qquad \widehat{\vec{J}}_X{}^0 = z^0\vec{A}_X,$$
$$\widehat{\vec{J}}_0{}^\beta = \tfrac{1}{z^0}\vec{k}^\beta, \qquad \widehat{\vec{J}}_\alpha{}^\beta = \tfrac{1}{z^0}\vec{k}_\alpha \times \vec{k}^\beta, \qquad \widehat{\vec{J}}_X{}^\beta = \vec{A}_X \times \vec{k}^\beta + \vec{J}_X{}^Z(\vec{A}_Z \cdot \vec{k}^\beta), \tag{3.12}$$
$$\widehat{\vec{J}}_0{}^Y = 0, \qquad \widehat{\vec{J}}_\alpha{}^Y = 0, \qquad \widehat{\vec{J}}_X{}^Y = \vec{J}_X{}^Y.$$

Finally, for the $USp(2,2n_H)$ connections we find the following nonzero components:

$$\hat{\omega}_{0A}{}^B = \tfrac{1}{2}f_Y^{iB}\partial_0 f_{iA}^Y + \tfrac{1}{2z^0}\delta_A^B,$$
$$\hat{\omega}_{\alpha i}{}^j = -\mathrm{i}\tfrac{1}{2z^0}\vec{k}_\alpha \cdot \vec{\sigma}_i{}^j, \qquad \hat{\omega}_{\alpha A}{}^B = \tfrac{1}{2}f_Y^{iB}\partial_\alpha f_{iA}^Y,$$
$$\hat{\omega}_{Xi}{}^j = A_X^\alpha \hat{\omega}_{\alpha i}{}^j = \omega_{Xi}{}^j, \qquad \hat{\omega}_{XA}{}^B = \omega_{XA}{}^B, \tag{3.13}$$
$$\hat{\omega}_{Xi}{}^A = \mathrm{i}\sqrt{\tfrac{1}{2z^0}}\varepsilon_{ik}f_X^{kA}, \qquad \hat{\omega}_{XA}{}^i = -\mathrm{i}\sqrt{\tfrac{z^0}{2}}\varepsilon^{ij}f_{jA}^Y h_{YX},$$

where $\omega_{XA}{}^B$ is defined in terms of the quaternionic-Kähler vielbeins f_X^{iA}, the Levi-Civita connection computed with $z^0 h_{XY}$ and the $SU(2)$ connection $\omega_{Xi}{}^j$ on the quaternionic-Kähler manifold through the covariant constancy of the vielbeins.[4] Using the vector notation the $SU(2)$ connection is given by

$$\vec{\omega}_X = -\tfrac{1}{2}\vec{A}_X. \tag{3.14}$$

Using the above results, one can express any hatted quantity (geometric quantity of the hyper-Kähler space) in terms of the unhatted ones (geometric quantity of the quaternionic-Kähler space) and vice versa. Here we list some explicit expressions for hatted quantities

---

[3]We consider here the domain $z^0 > 0$, which is necessary to obtain at the end positive kinetic terms for the graviton.

[4]Note the subtle difference in notation: in the hyper-Kähler manifold there is no $SU(2)$ connection, and $\hat{\omega}_{Xi}{}^j$ are components of the connection $\hat{\omega}_{X\hat{A}}{}^{\hat{B}}$. On the other hand, in the quaternionic space these components do not exist and $\omega_{Xi}{}^j = \mathrm{i}\vec{\omega}_X \cdot \vec{\sigma}_i{}^j$ is the $SU(2)$ connection.

that occur in the construction of the action,

$$\hat{C}_{AB} = C_{AB}, \qquad \hat{C}_{ij} = \varepsilon_{ij}, \qquad \hat{C}_{iA} = 0,$$
$$\widehat{\vec{P}}_I = \vec{P}_I,$$
$$\hat{k}_I^{\hat{X}} = \{0, -2\vec{k}^\alpha \cdot (\vec{\omega}_X k_I^X - \tfrac{1}{z^0}\vec{P}_I), k_I^X\},$$
$$\widehat{W}_{ABC}{}^D = \mathcal{W}_{ABC}{}^D. \tag{3.15}$$

Here, $\vec{P}_I$ are the moment maps of the quaternionic-Kähler manifold, in principle to be distinguished from the hyper-Kähler moment maps defined in (2.18) (all hyper-Kähler relations from that section should here be interpreted with hatted quantities and indices). For $n_H \neq 0$ they are defined in terms of the Killing vectors and complex structures as

$$4n_H \vec{P}_I = z_0 \vec{J}_X{}^Y \mathfrak{D}_Y k_I^X. \tag{3.16}$$

In the absence of any physical hypermultiplets, i.e. $n_H = 0$, moment maps can still be present. In fact, they are constants, 'Fayet-Iliopoulos (FI) terms', restricted by the 'equivariance condition'

$$2\nu \vec{P}_I \times \vec{P}_J + f_{IJ}{}^K \vec{P}_K = 0, \tag{3.17}$$

where ν is not determined (multiplying again with ν, this becomes an equation for $\nu \vec{P}_I$). This will be further analysed in section 5. The moment maps (3.16) satisfy a similar equivariance condition [33]. Equations (3.15) say that the hyper-Kähler moment maps are the same as these moment maps, both when $n_H \neq 0$ and when $n_H = 0$. In the latter case, the FI terms are therefore related to the gauge transformation of the scalars of the compensating hypermultiplet, as is the case in 4 dimensions [34].

Similar to the W_{ABCD} tensor (2.14) on a hyper-Kähler manifold, we can introduce $\mathcal{W}_{ABCD}$, which is a completely symmetric and traceless tensor. It is defined by the quaternionic-Kähler Riemann tensor $R_{XYZ}{}^W$, and conversely it determines the latter, as follows:

$$\mathcal{W}_{AB}{}^{CD} = \tfrac{1}{2} L^{XY}{}_{BA} L_{WZ}{}^{CD} R_{XY}{}^{ZW} + 2\frac{1}{z^0} \delta_{(A}{}^C \delta_{B)}{}^D, \qquad L^{XY}{}_{BA} \equiv f_A^{iX} f_{iB}^Y, \tag{3.18}$$

$$2R_{XYWZ} = -\frac{1}{z^0} \left(g_{Z[X} g_{Y]W} + \vec{J}_{XY} \cdot \vec{J}_{ZW} - \vec{J}_{Z[X} \cdot \vec{J}_{Y]W} \right) + L_{ZW}{}^{AB} \mathcal{W}_{ABCD} L_{XY}{}^{CD}.$$

We raise and lower A, B indices using C_{AB} and X, Y indices using g_{XY}, see (3.6).

For every point in $\{z^0, z^\alpha\}$, the $\{q^X\}$ subspace describes a quaternionic-Kähler manifold. These quaternionic-Kähler manifolds are related by coordinate redefinitions, $SU(2)$ gauge transformations and/or dilatations. Note that before gauge-fixing, all unhatted objects *a priori* are still dependent on both z^0, z^α and q^X. For every gauge-fixing, eliminating the compensator fields z^0, z^α in terms of constants (or functions of q^X), they become geometrical objects on the quaternionic-Kähler manifold.

Note that in this section we only discussed the geometry related to the hypermultiplets. The (very special) geometry related to the vector-multiplets will be discussed in the next section when we perform the gauge-fixing. The reason for this is that the discussion of the vector-multiplet geometries requires the use of the equations of motion and therefore cannot be discussed independently of the physical theory.

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4 Gauge-fixing

The actions given in appendix B are invariant under the full superconformal group. In order to break the symmetries that are not present in the Poincaré algebra, we will impose the necessary gauge conditions in the following subsections. This is analogous to the corresponding steps in $N = 2$ supergravity in $D = 4$ [35] and in $D = 6$ [28].

Before carrying out all technicalities implied by the gauge-fixing procedure, it is instructive to outline the steps we are going to follow. Just like in the example of section 1 the extra (superconformal) symmetries can be removed with the help of compensating multiplets. In our particular case, one hypermultiplet together with one vector multiplet will play the role of compensating multiplets. Our strategy is illustrated in figure 1. In this figure we have summarized which fields are eliminated by gauge-fixing and/or solving the equations of motion[5].

The field content of the matter-coupled conformal supergravity is given by the Weyl multiplet, $n_H + 1$ hypermultiplets and $n_V + 1$ Yang-Mills vector multiplets combined with n_T tensor multiplets in a vector-tensor multiplet. Note that in figure 1 we represented only the independent fields of the Weyl multiplet. The dependent fields $f_\mu{}^a$, $\hat{\omega}_\mu{}^{ab}$ and ϕ_μ^i are expressed in terms of the former in the first step (see below). Note that the b_μ field does not enter the action; it can therefore be set to zero as a gauge condition for the special conformal (or K-)transformations. There exist several auxiliary fields both in the standard Weyl multiplet (V_μ^{ij} and T_{ab}) as in the vector multiplets (Y^{ijI}). They are eliminated by solving the corresponding field equations.

The equations of motion of the χ^i and D fields collaborate with the S-gauge and D-gauge, respectively, to remove the fermionic degrees of freedom of the compensating multiplets and two of their scalars, i.e. σ and z^0. We see from the figure that all field components of the compensating hypermultiplet are eliminated by the gauge-fixing procedure. The only field component of the compensating vector multiplet surviving the gauge-fixing procedure is the gauge potential A_μ, which contributes to the graviphoton field in the Poincaré multiplet. As we eliminated the auxiliary fields Y^{ij}, the vector multiplets, as well as their tensor multiplet companions, are realized on-shell in the Poincaré theory. We thus end up with a matter-coupled Poincaré supergravity theory containing, besides the Poincaré multiplet, n_V vector, n_T tensor and n_H hypermultiplets. The geometry described by the moduli of the latter modifies during the gauge-fixing according to our discussion in section 3. We will see below that the vector scalars also parametrize a particular type of manifold at the Poincaré level, namely a very special real manifold (see section 4.3).

---

[5]The arrows show that one can imagine another construction. Namely we may obtain conformally invariant actions with consistent field equations by only using a compensating vector multiplet and no compensating hypermultiplet. The compensating vector multiplet is necessary to obtain consistent field equations for the auxiliary D and χ_i. However, the compensating hypermultiplet is only needed to break conformal invariance.
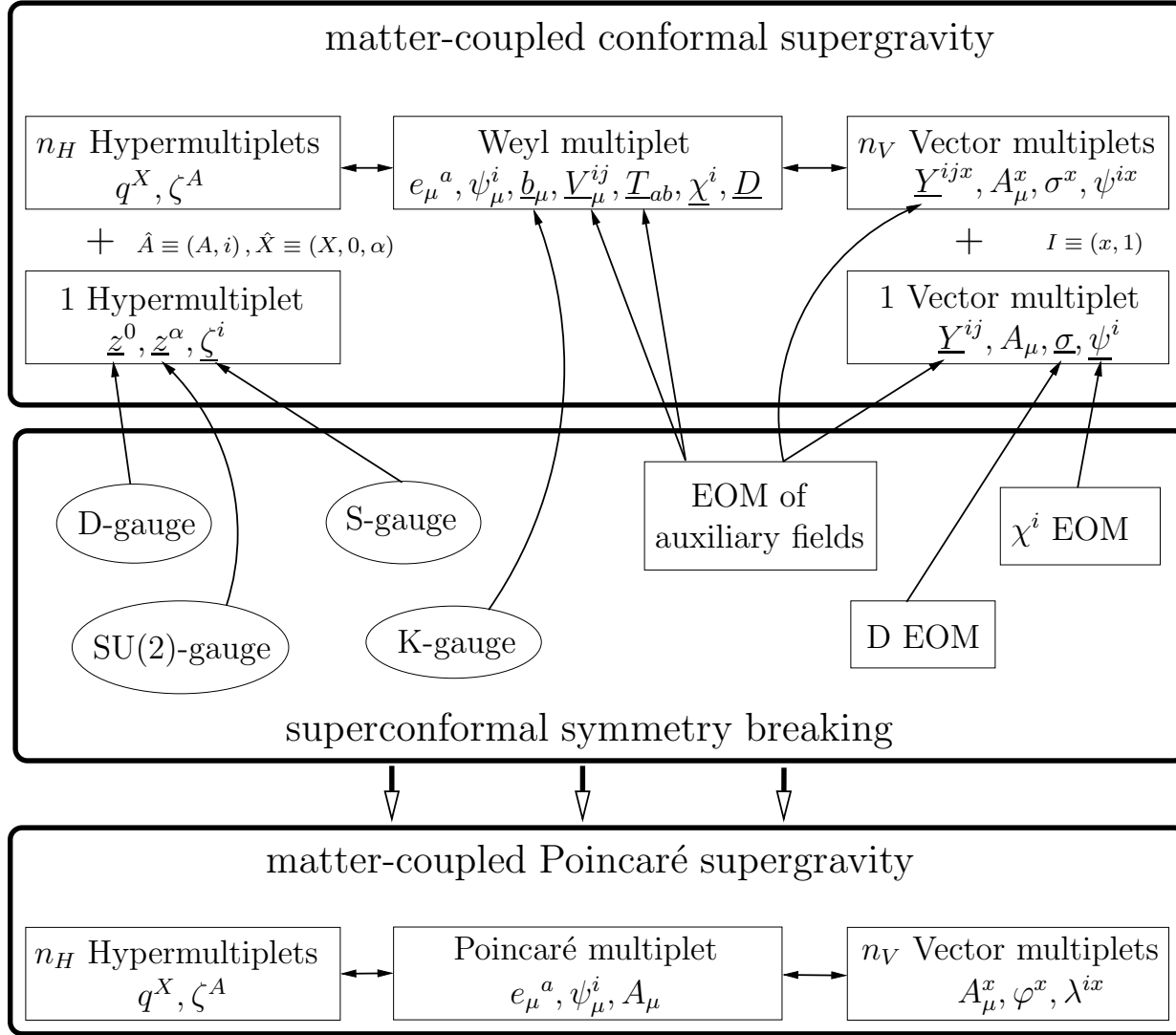
Figure 1: *The gauge-fixing procedure: the underlined fields are eliminated when passing to Poincaré SUGRA. The arrows indicate how these fields are eliminated: by gauge-fixing a symmetry or by applying an equation of motion.*

One should keep in mind that during the gauge-fixing procedure the definition of the covariant derivatives changes. Indeed, when passing from a superconformal invariant theory to a super-Poincaré theory, the remaining fields are chosen such that they do not transform under the broken symmetries, e.g. the scale symmetries. These scale symmetries generated terms in the superconformal covariant derivative that are absent in the Poincaré covariant derivatives. Another thing to keep in mind has to do with the transformation rules. In the conformally invariant theory, these transformation rules involve the parameters of the conformal transformations. Due to the gauge-fixing conditions, these parameters become dependent and are expressed in terms of the parameters of the Poincaré theory through the so-called decomposition rules. The super-Poincaré transformation rules are therefore inferred from the superconformal ones after eliminating auxiliary fields and employing the decomposition rules. This finishes our overview of the gauge-fixing procedure. We now proceed with a more technical discussion of the same procedure.

4.1 Preliminaries

The first step in the gauge-fixing process is the elimination of the dependent gauge fields ϕ_μ^i and $f_\mu{}^a$, associated with S- and K-symmetries, respectively. Using the relations given in [18, (3.11)] together with the definitions of the supercovariant curvatures, we find the following expressions for these gauge fields:

$$
\begin{aligned}
f_a{}^a &= \tfrac{1}{16}\left(-R(\hat\omega) - \tfrac{1}{3}\bar\psi_\rho\gamma^{\rho\mu\nu}\mathcal{D}_\mu\psi_\nu\right.\\
&\quad \left. + \tfrac{1}{3}\bar\psi_a^i\gamma^{abc}\psi_b^j V_{cij} + 16\bar\psi_a\gamma^a\chi - 4\mathrm{i}\bar\psi^a\psi^b T_{ab} + \tfrac{4}{3}\mathrm{i}\bar\psi^b\gamma_{abcd}\psi^a T^{cd}\right),\\
\hat\omega_\mu{}^{ab} &= \omega_\mu{}^{ab}(e) - \tfrac{1}{2}\bar\psi^{[b}\gamma^{a]}\psi_\mu - \tfrac{1}{4}\bar\psi^b\gamma_\mu\psi^a + 2e_\mu{}^{[a}b^{b]},\\
\phi_\mu^i &= \tfrac{1}{2}\mathrm{i}\gamma^\nu\mathcal{D}_{[\mu}\psi_{\nu]}^i - \tfrac{1}{12}\mathrm{i}\gamma_\mu{}^{\nu\rho}\mathcal{D}_\nu\psi_\rho^i - \tfrac{1}{2}\mathrm{i}V_{[\mu}{}^{ij}\gamma^\nu\psi_{\nu]j} + \tfrac{1}{12}\mathrm{i}V_a{}^{ij}\gamma_\mu{}^{ab}\psi_{bj}\\
&\quad - T^a{}_\mu\psi_a^i - \tfrac{1}{3}T^{ab}\gamma_{b\mu}\psi_a^i - \tfrac{2}{3}T_{b\mu}\gamma^{ab}\psi_a^i - \tfrac{1}{3}T_{bc}\gamma^{abc}{}_\mu\psi_a^i\\
&\quad - \tfrac{1}{12}\mathrm{i}(\gamma^{ab}\gamma_\mu - \tfrac{1}{2}\gamma_\mu\gamma^{ab})b_a\psi_b^i,
\end{aligned}
\tag{4.1}
$$

with

$$
\mathcal{D}_\mu = \partial_\mu + \tfrac{1}{4}\hat\omega_\mu{}^{ab}\gamma_{ab}.
\tag{4.2}
$$

We only need the contracted version of $f_\mu{}^a$ since the other components do not appear in the action or transformation rules. In order to simplify the notation we choose to work with $\hat\omega_\mu{}^{ab}$ instead of $\omega_\mu{}^{ab}$ for the time being.

After writing out all covariant derivatives and dependent gauge fields, the gauge field b_μ does not appear in the action. This can be understood from the special conformal symmetry (or K-symmetry) of the action. We will choose the conventional gauge choice for the K-invariance, namely

$$
K\text{-gauge:}\quad b_\mu = 0.
\tag{4.3}
$$

At this point we are left with one more gauge field corresponding to a non-Poincaré symmetry: the $SU(2)$ gauge field V_μ^{ij}. Solving for its equation of motion, corresponding to

the action (1.6), yields the following expression:

$$
\begin{aligned}
V_\mu^{ij} &= \frac{9}{2k^2}\Big(\hat{g}_{\hat{X}\hat{Y}}\big(\partial_\mu q^{\hat{X}} + gA_\mu^I \hat{k}_I^{\hat{X}}\big)\hat{k}^{ij\hat{Y}} + \frac{1}{2}\,\mathrm{i}\hat{k}^{\hat{X}}\hat{f}_{\hat{X}}^{(i\hat{A}}\bar{\zeta}_{\hat{A}}\gamma_{\mu\nu}\psi^{\nu j)} - \mathrm{i}\hat{k}^{ij\hat{X}}\hat{f}_{k\hat{X}}^{\hat{A}}\bar{\zeta}_{\hat{A}}\gamma_\nu\gamma_\mu\psi^{k\nu} \\
&\quad -\tfrac{1}{2}C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{K}}\bar{\psi}^{i\widetilde{I}}\gamma_\mu\psi^{j\widetilde{J}} + \frac{1}{4}\,\mathrm{i}C_{\widetilde{I}J\widetilde{K}}\sigma^{\widetilde{K}}\sigma^{\widetilde{I}}\bar{\psi}^{i\widetilde{J}}\gamma_{\mu\nu}\psi^{j\nu}\Big).
\end{aligned}
\tag{4.4}
$$

The action further contains four auxiliary matter fields: D, T_{ab} and χ^i from the Weyl multiplet, and $Y_{ij}^{\widetilde{I}}$ from the vector-tensor multiplet. Both D and χ^i appear as Lagrange multipliers in the action, leading to the following constraints,

$$
D \;:\; \qquad C - \tfrac{1}{3}k^2 = 0, \qquad \text{with} \quad C \equiv C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}, \quad k^2 = \hat{k}^{\hat{X}}g_{\hat{X}\hat{Y}}\hat{k}^{\hat{Y}} = -9z^0, \tag{4.5}
$$

$$
\chi^i \;:\; \qquad -8\mathrm{i}C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\psi_i^{\widetilde{K}} - \tfrac{4}{3}\big(C - \tfrac{1}{3}k^2\big)\gamma^\mu\psi_{\mu i} + \tfrac{16}{3}\mathrm{i}A_i^{\hat{A}}\zeta_{\hat{A}} = 0. \tag{4.6}
$$

Here, and for later purposes, we have introduced sections $A_i^{\hat{A}}$ that are defined by [24]

$$
A_i^{\hat{A}} = \hat{k}^{\hat{X}}f_{i\hat{X}}^{\hat{A}}. \tag{4.7}
$$

The equations of motion for $Y_{ij}^{\widetilde{I}}$ and T_{ab} are given by

$$
Y^{ij\widetilde{J}}C_{\widetilde{I}\widetilde{J}\widetilde{K}}\sigma^{\widetilde{K}} = -g\delta_{\widetilde{I}}^L \hat{P}_L^{ij} + \tfrac{1}{4}\mathrm{i}C_{\widetilde{I}\widetilde{J}\widetilde{K}}\bar{\psi}^{i\widetilde{J}}\psi^{j\widetilde{K}}, \tag{4.8}
$$

$$
\begin{aligned}
T_{ab} &= \frac{9}{64k^2}\Big(4\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\widehat{\mathcal{H}}_{ab}^{\widetilde{K}}C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}^{\widetilde{K}}\gamma_{[a}\psi_{b]}C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar{\psi}^{\widetilde{K}}\gamma_{abc}\psi^c C_{\widetilde{I}\widetilde{J}\widetilde{K}} \\
&\quad + \mathrm{i}\sigma^{\widetilde{I}}\bar{\psi}^{\widetilde{J}}\gamma_{ab}\psi^{\widetilde{K}}C_{\widetilde{I}\widetilde{J}\widetilde{K}} + \tfrac{2}{3}\hat{k}^{\hat{X}}\hat{f}_{i\hat{X}}^{\hat{A}}\bar{\zeta}_{\hat{A}}\gamma_{[a}\psi_{b]}^i + \tfrac{2}{3}\hat{k}^{\hat{X}}\hat{f}_{i\hat{X}}^{\hat{A}}\bar{\zeta}_{\hat{A}}\gamma_{abc}\psi^{ic} + 2\mathrm{i}\bar{\zeta}_{\hat{A}}\gamma_{ab}\zeta^{\hat{A}}\Big).
\end{aligned}
\tag{4.9}
$$

These equations have been simplified by using (4.5).

4.2 Gauge choices and decomposition rules

Apart from the K-gauge (4.3) that we have already introduced to fix the special conformal symmetry, we now choose gauges for the other non-Poincaré (super)symmetries as well.

D-gauge. Demanding canonical factors for the Einstein-Hilbert and Rarita-Schwinger kinetic terms in (1.6), we have to impose the following D-gauge:

$$
D\text{-gauge:} \quad \frac{1}{24}\big(C + k^2\big) = -\frac{1}{2\kappa^2}. \tag{4.10}
$$

where κ has dimensions of $[\text{length}]^{3/2}$. If we combine the D-gauge (4.10) and D-EOM (4.5) we obtain

$$
k^2 = -\frac{9}{\kappa^2}, \qquad C = -\frac{3}{\kappa^2}. \tag{4.11}
$$

In the light of (3.2)-(3.4) the first constraint implies that

$$
z^0 = \kappa^{-2}, \tag{4.12}
$$

whereas the second constraint effectively eliminates one of the vector-tensor scalars[6].

S-gauge. In the action (1.6) there appear terms where $\gamma^\mu \phi_\mu = \frac{1}{4} i \gamma^{\mu\nu} \partial_\mu \psi_\nu +$ (non-derivative terms) is multiplied by hyperino and gaugino fields. These terms imply a mixing of the kinetic terms of the gravitino with the hyperino and gaugino fields. A suitable S-gauge can eliminate this mixing: we put the coefficient of the above expression equal to zero:

$$S\text{-gauge:}\quad C_{\widetilde{I}\widetilde{J}\widetilde{K}} \sigma^{\widetilde{I}} \sigma^{\widetilde{J}} \psi_i^{\widetilde{K}} + 2 A_i^{\hat{A}} \zeta_{\hat{A}} = 0. \tag{4.13}$$

Combining this with the χ field equation (4.6) leads to

$$C_{\widetilde{I}\widetilde{J}\widetilde{K}} \sigma^{\widetilde{I}} \sigma^{\widetilde{J}} \psi_i^{\widetilde{K}} = 0, \qquad A_i^{\hat{A}} \zeta_{\hat{A}} = 0. \tag{4.14}$$

In our coordinate basis, we obtain the following expression for the sections $A_{\hat{A}}^i$:

$$A_{\hat{A}}^i \equiv \varepsilon^{ij} \hat{k}_{\hat{X}} \hat{f}_{j\hat{A}}^{\hat{X}} = -3 \varepsilon^{ij} \hat{f}_{j\hat{A}}^0 = -3 i \sqrt{\frac{z^0}{2}} \delta_{\hat{A}}^i. \tag{4.15}$$

Therefore, our choice of coordinates on the hyper-Kähler manifold is consistent with the fact that the hyperinos of the compensating hypermultiplet carry no physical degree of freedom:

$$\zeta^i = 0. \tag{4.16}$$

$SU(2)$**-gauge.** The gauge for dilatations was chosen such that $z^0 = \kappa^{-2}$. Similarly we may also choose a gauge for $SU(2)$. Such a gauge would be a specific point in the 3-dimensional space of the z^α. In principle, we could thus choose $z^\alpha = z_0^\alpha(q)$ for any function $z_0^\alpha(q)$, but we will restrict ourselves here to constants z_0^α:

$$SU(2)\text{-gauge:}\quad z^\alpha = z_0^\alpha. \tag{4.17}$$

Decomposition rules. As a consequence of the gauge choices, the corresponding transformation parameters can be expressed in terms of the independent ones by the so-called decomposition rules. For example, the requirement that the K-gauge (4.3) should be invariant under the most general superconformal transformation, i.e. $\delta b_\mu = 0$, leads to the following decomposition rule for Λ_K^a:

$$\Lambda_K^a = -\tfrac{1}{2} e^{\mu a} \left(\partial_\mu \Lambda_D + \tfrac{1}{2} i \bar{\epsilon} \phi_\mu - 2 \bar{\epsilon} \gamma_\mu \chi + \tfrac{1}{2} i \bar{\eta} \psi_\mu \right). \tag{4.18}$$

Similarly, demanding $\delta z^0 = 0$ yields

$$\Lambda_D = 0. \tag{4.19}$$

The decomposition rule for η^i can be found by varying the S-gauge and demanding that

$$\delta \left(C_{\widetilde{I}\widetilde{J}\widetilde{K}} \sigma^{\widetilde{I}} \sigma^{\widetilde{J}} \psi^{i\widetilde{K}} \right) = 0. \tag{4.20}$$

---

[6]The constraint (4.12) implies that the parameter ν defined in [19] is given by $\nu = -\kappa^2$. This parameter also appeared in (3.17) but from now on will not appear anymore in this paper.

We find

$$\kappa^{-2}\eta^i = -\tfrac{1}{12}C_{\tilde{I}\tilde{J}\tilde{K}}\sigma^{\tilde{I}}\sigma^{\tilde{J}}\gamma\cdot\widehat{\mathcal{H}}^{\tilde{K}}\epsilon^i + \tfrac{1}{3}g\sigma^I P_I^{ij}\epsilon_j + \tfrac{1}{32}\,\mathrm{i}\gamma^{ab}\epsilon^i\bar{\zeta}_A\gamma_{ab}\zeta^A$$
$$+ \tfrac{1}{16}\,\mathrm{i}C_{\tilde{I}\tilde{J}\tilde{K}}\sigma^{\tilde{I}}\Big(\gamma^a\epsilon_j\bar{\psi}^{i\tilde{J}}\gamma_a\psi^{j\tilde{K}} - \tfrac{1}{16}\gamma^{ab}\epsilon^i\bar{\psi}^{\tilde{J}}\gamma_{ab}\psi^{\tilde{K}}\Big). \tag{4.21}$$

The $SU(2)$ decomposition rule can be found by requiring that $\delta z^\alpha = 0$:

$$\vec{\Lambda}_{SU(2)} = \vec{\omega}_X(\delta_Q + \delta_G)q^X + g\Lambda_G^I\vec{P}_I. \tag{4.22}$$

4.3 Hypersurfaces

We now discuss the geometry for the vector-multiplet scalars. These arise as a consequence of the gauge-fixing procedure. In order to get a standard normalization, we rescale the $C_{\tilde{I}\tilde{J}\tilde{K}}$ symbol and the vector multiplet scalars as follows:

$$\sigma^{\tilde{I}} \equiv \sqrt{\frac{3}{2\kappa^2}}h^{\tilde{I}}, \qquad C_{\tilde{I}\tilde{J}\tilde{K}} \equiv -2\sqrt{\frac{2\kappa^2}{3}}\mathcal{C}_{\tilde{I}\tilde{J}\tilde{K}}, \tag{4.23}$$

such that

$$\mathcal{C}_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{I}}h^{\tilde{J}}h^{\tilde{K}} = 1. \tag{4.24}$$

The constraint (4.24) defines an $(n_V + n_T)$-dimensional hypersurface of scalars ϕ^x called a 'very special real' manifold, embedded into a $(n_V + n_T + 1)$-dimensional space spanned by the scalars $h^{\tilde{I}}(\phi)$.

The metric on the embedding $h^{\tilde{I}}$-manifold can be determined by substituting the equation of motion for T_{ab} (4.9) back into the action, and defining the kinetic term for the vectors/tensors as

$$\mathcal{L}_{\text{kin, Vector}-\text{Tensor}} = -\tfrac{1}{4}a_{\tilde{I}\tilde{J}}\widehat{\mathcal{H}}_{\mu\nu}^{\tilde{I}}\widehat{\mathcal{H}}^{\mu\nu\tilde{J}}. \tag{4.25}$$

We find

$$a_{\tilde{I}\tilde{J}} = -2\mathcal{C}_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{K}} + 3h_{\tilde{I}}h_{\tilde{J}}, \tag{4.26}$$

where

$$h_{\tilde{I}} \equiv a_{\tilde{I}\tilde{J}}h^{\tilde{J}} = \mathcal{C}_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{J}}h^{\tilde{K}} \qquad \Rightarrow \qquad h_{\tilde{I}}h^{\tilde{I}} = 1. \tag{4.27}$$

In the following we will assume that $a_{\tilde{I}\tilde{J}}$ is invertible; this enables us to solve (4.8) for $Y^{\tilde{I}ij}$. Following [21], we introduce

$$h_x^{\tilde{I}} \equiv -\sqrt{\frac{3}{2\kappa^2}}h_{,x}^{\tilde{I}}(\phi) \qquad \rightarrow \qquad h_{\tilde{I}x} \equiv a_{\tilde{I}\tilde{J}}h_x^{\tilde{J}}(\phi) = \sqrt{\frac{3}{2\kappa^2}}h_{\tilde{I},x}(\phi). \tag{4.28}$$

The metric on the manifold spanned by the scalars ϕ^x is the pull-back on the hypersurface of the metric $a_{\tilde{I}\tilde{J}}$ on the embedding space, i.e.

$$g_{xy} = h_x^{\tilde{I}}h_y^{\tilde{J}}a_{\tilde{I}\tilde{J}}. \tag{4.29}$$

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We furthermore define

$$h_{\tilde{I}}^x \equiv g^{xy}h_{\tilde{I}y}. \tag{4.30}$$

Several useful identities that were already discovered in [21] are summarized in appendix C.

The gauginos $\psi^{\tilde{I}}$ are constrained fields, due to the S-gauge. In order to translate these to $(n_V + n_T)$ unconstrained gauginos, we introduce $\lambda^{i\,x}$, which transform as vectors in the tangent space on the hypersurface. As we will see later, a convenient choice is given by (for agreement with the literature [2])[7]

$$\lambda^{ix} \equiv -h_{\tilde{I}}^x\psi^{i\tilde{I}}, \qquad \psi^{i\tilde{I}} = -h_x^{\tilde{I}}\lambda^{ix}. \tag{4.31}$$

Note that this choice for $\psi^{i\tilde{I}}$ indeed solves the S-gauge (4.14).

5 Results

After applying the steps outlined in the previous section, i.e. using a special coordinate basis, substituting the expressions for the dependent gauge fields and matter fields and 'reducing' the objects on the hyper-Kähler manifold to the quaternionic-Kähler manifold, we obtain the $N = 2$ super-Poincaré action.

We give in this section the full action for a number of vector multiplets (indices I), tensor multiplets (indices M, and together with the vector multiplets indicated as $\tilde{I}$, and their unconstrained fields with x) and hypermultiplets (indices X for the scalars and A for the spinors). The couplings of the vector and tensor multiplets are determined by the constants $\mathcal{C}_{\tilde{I}\tilde{J}\tilde{K}}$, a symplectic metric Ω_{MN} and the transformation matrices $t_{I\tilde{J}}{}^{\tilde{K}}$ related by (2.5), see also (2.3). The related quantities are defined in section 4.3.

We define the supercovariant field strengths $\widehat{F}_{ab}^I$ and a tensor field $\widetilde{B}_{ab}^M$ such that

$$\widehat{\mathcal{H}}_{ab}^{\tilde{I}} = \left(\widehat{F}_{ab}^I, B_{ab}^M\right) = \left(F_{ab}^I - \bar{\psi}_{[a}\gamma_{b]}\psi^I + \frac{\sqrt{6}}{4\kappa}\,\mathrm{i}\bar{\psi}_a\psi_b h^I, \widetilde{B}_{ab}^M - \bar{\psi}_{[a}\gamma_{b]}\psi^M + \frac{\sqrt{6}}{4\kappa}\,\mathrm{i}\bar{\psi}_a\psi_b h^M\right),$$

$$H_{ab}^{\tilde{I}} \equiv \left(F_{ab}^I, \widetilde{B}_{ab}^M\right), \qquad F_{\mu\nu}^I \equiv 2\partial_{[\mu}A_{\nu]}^I + gf_{JK}{}^I A_\mu^J A_\nu^K. \tag{5.1}$$

The B_{ab}^M transforms covariantly, as does $\hat{F}_{ab}$, while the action gets a simpler form using F_{ab} and $\widetilde{B}_{ab}$.

The hypermultiplets are completely characterized in terms of the vielbeins f_X^{iA}, that determine complex structures, $USp(2n_H)$ and $SU(2)$ connections. They transform in general under the gauge group of the vector multiplets. The Killing vectors k_I^X determine $t_{IA}{}^B$ and are restricted by (2.12). They determine the moment maps by (3.16). As mentioned in section 3, the moment maps can also exist without a quaternionic-Kähler manifold ($n_H = 0$), in which case they are the constant 'FI terms'. These are possible for two cases. First, in the case where the gauge group contains an $SU(2)$ factor, we can have

$$\vec{P}_I = \vec{e}_I\xi, \tag{5.2}$$

---

[7]We avoid here the introduction of a local basis for the fermions indicated by indices $\tilde{a}$ in [2].

where ξ is an arbitrary constant, and $\vec{e}_I$ are constants that are nonzero only for I in the range of the $SU(2)$ factor and satisfy

$$\vec{e}_I \times \vec{e}_J = f_{IJ}{}^K \vec{e}_K, \tag{5.3}$$

in order that (3.17) is verified.

The second case is $U(1)$ FI terms. In this case

$$\vec{P}_I = \vec{e}\,\xi_I, \tag{5.4}$$

where $\vec{e}$ is an arbitrary vector in $SU(2)$ space and ξ_I are constants for the I corresponding to $U(1)$ factors in the gauge group.

To be able to write down the potential and the supersymmetry transformation rules in an elegant fashion, we define

$$
\begin{aligned}
W^x &\equiv \tfrac{\sqrt{6}}{4}\kappa h^I K_I^x = -\tfrac{3}{4}t_{J\widetilde{I}}{}^{\widetilde{P}}h^J h^{\widetilde{I}}h_{\widetilde{P}}^x, && K_I^x \equiv -\tfrac{1}{\kappa}\sqrt{\tfrac{3}{2}}t_{I\widetilde{J}}{}^{\widetilde{K}}h^{\widetilde{J}}h_{\widetilde{K}}^x = -\tfrac{1}{\kappa}\sqrt{\tfrac{3}{2}}t_{I\widetilde{J}}{}^{\widetilde{K}}h^{\widetilde{J}x}h_{\widetilde{K}}, \\
\vec{P} &\equiv \kappa^2 h^I \vec{P}_I, && \vec{P}_x \equiv \kappa^2 h_x^I \vec{P}_I, \\
\mathcal{N}^{iA} &\equiv \tfrac{\sqrt{6}}{4}\kappa h^I k_I^X f_X^{iA}, && \\
T_{xyz} &\equiv \mathcal{C}_{\widetilde{I}\widetilde{J}\widetilde{K}}h_x^{\widetilde{I}}h_y^{\widetilde{J}}h_z^{\widetilde{K}}, && \Gamma_{xy}^w = h_{\widetilde{I}}^w h_{x,y}^{\widetilde{I}} + \kappa\sqrt{\tfrac{2}{3}}T_{xyz}g^{zw},
\end{aligned}
\tag{5.5}
$$

the latter being the Levi-Civita connection of g_{xy}.

The covariant derivatives now read

$$
\begin{aligned}
\mathcal{D}_\mu h^{\widetilde{I}} &= \partial_\mu h^{\widetilde{I}} + gt_{J\widetilde{K}}{}^{\widetilde{I}}A_\mu^J h^{\widetilde{K}} = -\sqrt{\tfrac{2}{3}}\kappa h_x^{\widetilde{I}}\left(\partial_\mu \phi^x + gK_J^x A_\mu^J\right) = -\sqrt{\tfrac{2}{3}}\kappa h_x^{\widetilde{I}}\mathcal{D}_\mu \phi^x, \\
\mathcal{D}_\mu q^X &= \partial_\mu q^X + gA_\mu^I k_I^X, \\
\mathcal{D}_\mu \lambda^{xi} &= \partial_\mu \lambda^{xi} + \partial_\mu \phi^y \Gamma_{yz}^x \lambda^{zi} + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\lambda^{xi} \\
&\quad + \partial_\mu q^X \omega_{Xj}{}^i \lambda^{xj} - g\kappa^2 A_\mu^I P_I^{ij}\lambda_j^x + gA_\mu^I K_I^{x;y}\lambda_y^i, \\
\mathcal{D}_\mu \zeta^A &= \partial_\mu \zeta^A + \partial_\mu q^X \omega_{XB}{}^A \zeta^B + \tfrac{1}{4}\omega_\mu{}^{bc}\gamma_{bc}\zeta^A + gA_\mu^I t_{IB}{}^A \zeta^B, \\
\mathcal{D}_\mu \psi_\nu^i &= \left(\partial_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\psi_\nu^i - \partial_\mu q^X \omega_X{}^{ij}\psi_{\nu j} - g\kappa^2 A_\mu^I P_I^{ij}\psi_{\nu j}.
\end{aligned}
\tag{5.6}
$$

Here $K_I^{x;y}$ stands for the covariant derivative, where the index is raised with the inverse metric g^{xy}. We choose to extract the fermionic terms from the spin connection, using $\omega_{\mu a}{}^b$ instead of $\hat{\omega}_{\mu a}{}^b$ in the covariant derivatives and the Ricci scalar, unless otherwise mentioned.

Performing all the steps of the conformal programme we find the following action:

$$
\begin{aligned}
e^{-1}\mathcal{L} &= \tfrac{1}{2\kappa^2}R(\omega) - \tfrac{1}{4}a_{\widetilde{I}\widetilde{J}}\widehat{\mathcal{H}}_{\mu\nu}^{\widetilde{I}}\widehat{\mathcal{H}}^{\widetilde{J}\mu\nu} - \tfrac{1}{2}g_{xy}\mathcal{D}_\mu \phi^x \mathcal{D}^\mu \phi^y - \tfrac{1}{2}g_{XY}\mathcal{D}_\mu q^X \mathcal{D}^\mu q^Y \\
&\quad + \frac{1}{16g}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}\widetilde{B}_{\mu\nu}^M\left(\partial_\rho \widetilde{B}_{\sigma\tau}^N + 2gt_{IJ}{}^N A_\rho^I F_{\sigma\tau}^J + gt_{IP}{}^N A_\rho^I \widetilde{B}_{\sigma\tau}^P\right) \\
&\quad - \tfrac{1}{2\kappa^2}\bar{\psi}_\rho \gamma^{\rho\mu\nu}\mathcal{D}_\mu \psi_\nu - \tfrac{1}{2}\bar{\lambda}_x \slashed{D}\lambda^x - \bar{\zeta}^A \slashed{D}\zeta_A \\
&\quad + \tfrac{g^2}{\kappa^4}\left(4\vec{P}\cdot\vec{P} - 2\vec{P}_x \cdot \vec{P}_x - 2W_x W^x - 2\mathcal{N}_{iA}\mathcal{N}^{iA}\right) \\
&\quad + \tfrac{\kappa}{6\sqrt{6}}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\mathcal{C}_{IJK}A_\mu^I\left[F_{\nu\lambda}^J F_{\rho\sigma}^K + f_{FG}{}^J A_\nu^F A_\lambda^G\left(-\tfrac{1}{2}gF_{\rho\sigma}^K + \tfrac{1}{10}g^2 f_{HL}{}^K A_\rho^H A_\sigma^L\right)\right]
\end{aligned}
$$

$$-\tfrac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^{M}t_{FG}{}^{N}A_{\mu}^{I}A_{\nu}^{F}A_{\lambda}^{G}\left(-\tfrac{1}{2}gF_{\rho\sigma}^{K}+\tfrac{1}{10}g^{2}f_{HL}{}^{K}A_{\rho}^{H}A_{\sigma}^{L}\right)$$

$$+\tfrac{1}{4}h_{\widetilde{I}x}H_{bc}^{\widetilde{I}}\bar{\psi}_{a}\gamma^{abc}\lambda^{x}-\tfrac{\sqrt{6}}{16\kappa}\mathrm{i}h_{\widetilde{I}}H^{cd\widetilde{I}}\bar{\psi}^{a}\gamma_{abcd}\psi^{b}+\tfrac{1}{4}\sqrt{\tfrac{2}{3}}\kappa\mathrm{i}\left(\tfrac{1}{4}g_{xy}h_{\widetilde{I}}+T_{xyz}h_{\widetilde{I}}^{z}\right)\lambda^{x}\gamma\cdot H^{\widetilde{I}}\lambda^{y}$$

$$+\tfrac{1}{8}\sqrt{6}\,\mathrm{i}\kappa h_{\widetilde{I}}\bar{\zeta}_{A}\gamma\cdot H^{\widetilde{I}}\zeta^{A}+\tfrac{1}{2}\mathrm{i}\bar{\psi}_{a}\slashed{D}\phi^{x}\gamma^{a}\lambda_{x}+\mathrm{i}\bar{\zeta}_{A}\gamma^{a}\slashed{D}q^{X}\psi_{a}^{i}f_{iX}^{A}$$

$$+g\Big[-\sqrt{\tfrac{3}{2}}\tfrac{1}{\kappa}\mathrm{i}h^{I}t_{IB}{}^{A}\bar{\zeta}_{A}\zeta^{B}+2\mathrm{i}k_{I}^{X}f_{iX}^{A}h_{x}^{I}\bar{\zeta}_{A}\lambda^{ix}$$

$$-\sqrt{\tfrac{2}{3}}\tfrac{1}{\kappa}\mathrm{i}\left(\tfrac{1}{4}g_{xy}P_{ij}+T_{xyz}P_{ij}^{z}\right)\bar{\lambda}^{ix}\lambda^{jy}+\tfrac{1}{\kappa}\bar{\lambda}^{x}\lambda^{y}\left(\tfrac{1}{2}\mathrm{i}h_{x}^{I}K_{Iy}-\tfrac{2}{3}\partial_{x}W_{y}\right)$$

$$-\tfrac{2}{\kappa^{2}}\mathcal{N}_{i}^{A}\bar{\zeta}_{A}\gamma^{a}\psi_{a}^{i}+\tfrac{1}{\kappa^{2}}\bar{\psi}_{a}^{i}\gamma^{a}\lambda^{jx}\left(P_{xij}+\varepsilon_{ij}W_{x}\right)+\sqrt{\tfrac{3}{8}}\tfrac{1}{\kappa^{3}}\mathrm{i}P_{ij}\bar{\psi}_{a}^{i}\gamma^{ab}\psi_{b}^{j}\Big]$$

$$-\tfrac{1}{32}\bar{\psi}_{a}^{i}\psi^{ja}\bar{\lambda}_{i}^{x}\lambda_{jx}-\tfrac{1}{32}\bar{\psi}_{a}^{i}\gamma_{b}\psi^{ja}\bar{\lambda}_{i}^{x}\gamma^{b}\lambda_{jx}-\tfrac{1}{128}\bar{\psi}_{a}\gamma_{bc}\psi^{a}\bar{\lambda}^{x}\gamma^{bc}\lambda_{x}$$

$$-\tfrac{1}{16}\bar{\psi}_{a}^{i}\gamma^{ab}\psi_{b}^{j}\bar{\lambda}_{i}^{x}\lambda_{jx}-\tfrac{1}{32}\bar{\psi}^{ai}\gamma^{bc}\psi^{dj}\bar{\lambda}_{i}^{x}\gamma_{abcd}\lambda_{jx}+\tfrac{1}{8\kappa^{2}}\bar{\psi}_{a}\gamma_{b}\psi^{b}\bar{\psi}^{a}\gamma_{c}\psi^{c}$$

$$-\tfrac{1}{16\kappa^{2}}\bar{\psi}_{a}\gamma_{b}\psi_{c}\bar{\psi}^{a}\gamma^{c}\psi^{b}-\tfrac{1}{32\kappa^{2}}\bar{\psi}_{a}\gamma_{b}\psi_{c}\bar{\psi}^{a}\gamma^{b}\psi^{c}+\tfrac{1}{32\kappa^{2}}\bar{\psi}_{a}\psi_{b}\bar{\psi}_{c}\gamma^{abcd}\psi_{d}$$

$$-\tfrac{1}{16}\bar{\psi}^{a}\gamma^{b}\psi^{c}\bar{\zeta}_{A}\gamma_{abc}\zeta^{A}+\tfrac{1}{16}\bar{\psi}_{a}\gamma^{bc}\psi^{a}\bar{\zeta}_{A}\gamma_{bc}\zeta^{A}-\tfrac{1}{16}\bar{\psi}^{a}\psi^{b}\bar{\zeta}_{A}\gamma_{ab}\zeta^{A}$$

$$+\tfrac{\kappa}{6}\sqrt{\tfrac{2}{3}}\mathrm{i}T_{xyz}\left(\bar{\psi}_{a}\gamma_{b}\lambda^{x}\bar{\lambda}^{y}\gamma^{ab}\lambda^{z}+\bar{\psi}_{a}^{i}\gamma^{a}\lambda^{jx}\bar{\lambda}_{i}^{y}\lambda_{j}^{z}\right)$$

$$+\tfrac{\kappa^{2}}{32}\bar{\lambda}^{x}\gamma_{ab}\lambda_{x}\bar{\zeta}_{A}\gamma^{ab}\zeta^{A}-\tfrac{\kappa^{2}}{16}\bar{\lambda}^{ix}\gamma_{a}\lambda_{x}^{j}\bar{\lambda}_{i}^{y}\gamma^{a}\lambda_{jy}$$

$$+\tfrac{\kappa^{2}}{128}\bar{\lambda}^{x}\gamma_{ab}\lambda_{x}\bar{\lambda}^{y}\gamma^{ab}\lambda_{y}+\tfrac{\kappa^{2}}{6}g^{zt}T_{xyz}T_{tvw}\bar{\lambda}^{ix}\lambda^{jy}\bar{\lambda}_{i}^{v}\lambda_{j}^{w}-\tfrac{\kappa^{2}}{48}\bar{\lambda}^{ix}\lambda_{x}^{j}\bar{\lambda}_{i}^{y}\lambda_{jy}$$

$$+\tfrac{\kappa^{2}}{32}\bar{\zeta}_{A}\gamma_{ab}\zeta^{A}\bar{\zeta}_{B}\gamma^{ab}\zeta^{B}-\tfrac{1}{4}\mathcal{W}_{ABCD}\bar{\zeta}^{A}\zeta^{B}\bar{\zeta}^{C}\zeta^{D}. \tag{5.7}$$

This action admits the following $N=2$ supersymmetry:

$$\delta e_{\mu}{}^{a}=\tfrac{1}{2}\bar{\epsilon}\gamma^{a}\psi_{\mu},$$

$$\delta\psi_{\mu}^{i}=D_{\mu}(\hat{\omega})\epsilon^{i}+\tfrac{\mathrm{i}\kappa}{4\sqrt{6}}h_{\widetilde{I}}\widehat{\mathcal{H}}^{\widetilde{I}\nu\rho}(\gamma_{\mu\nu\rho}-4g_{\mu\nu}\gamma_{\rho})\epsilon^{i}+\delta q^{X}\omega_{X}^{ij}\psi_{\mu j}-\tfrac{1}{\kappa\sqrt{6}}\mathrm{i}gP^{ij}\gamma_{\mu}\epsilon_{j}$$

$$-\tfrac{\kappa^{2}}{6}\epsilon_{j}\bar{\lambda}^{ix}\gamma_{\mu}\lambda_{x}^{j}+\tfrac{\kappa^{2}}{12}\gamma_{\mu\nu}\epsilon_{j}\bar{\lambda}^{ix}\gamma^{\nu}\lambda_{x}^{j}-\tfrac{\kappa^{2}}{48}\gamma_{\mu\nu\rho}\epsilon_{j}\bar{\lambda}^{ix}\gamma^{\nu\rho}\lambda_{x}^{j}+\tfrac{\kappa^{2}}{12}\gamma^{\nu}\epsilon_{j}\bar{\lambda}^{ix}\gamma_{\mu\nu}\lambda_{x}^{j}$$

$$+\tfrac{\kappa^{2}}{16}\gamma_{\mu\nu\rho}\epsilon^{i}\bar{\zeta}_{A}\gamma^{\nu\rho}\zeta^{A},$$

$$\delta h^{\widetilde{I}}=-\tfrac{\kappa}{\sqrt{6}}\mathrm{i}\bar{\epsilon}\lambda^{x}h_{x}^{\widetilde{I}}\quad,\qquad\delta\phi^{x}=\tfrac{1}{2}\mathrm{i}\bar{\epsilon}\lambda^{x},$$

$$\delta A_{\mu}^{I}=\vartheta_{\mu}^{I},$$

$$\delta\widetilde{B}_{\mu\nu}^{M}=2\mathcal{D}_{[\mu}\vartheta_{\nu]}^{M}-\sqrt{6}\tfrac{g}{\kappa}\bar{\epsilon}\gamma_{[\mu}\psi_{\nu]}h_{N}\Omega^{MN}-\mathrm{i}g\bar{\epsilon}\gamma_{\mu\nu}\lambda^{x}h_{xN}\Omega^{MN},$$

$$\delta\lambda^{xi}=-\tfrac{\mathrm{i}}{2}\widehat{\slashed{D}}\phi^{x}\epsilon^{i}-\delta\phi^{y}\Gamma_{yz}^{x}\lambda^{zi}+\delta q^{X}\omega_{X}{}^{ij}\lambda_{j}^{x}+\tfrac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{I}}h_{\widetilde{I}}^{x}\epsilon^{i}$$

$$\tfrac{\kappa}{4\sqrt{6}}T^{xyz}\left[3\epsilon_{j}\bar{\lambda}_{y}^{i}\lambda_{z}^{j}-\gamma^{\mu}\epsilon_{j}\bar{\lambda}_{y}^{i}\gamma_{\mu}\lambda_{z}^{j}-\tfrac{1}{2}\gamma^{\mu\nu}\epsilon_{j}\bar{\lambda}_{y}^{i}\gamma_{\mu\nu}\lambda_{z}^{j}\right]-\tfrac{1}{\kappa^{2}}gP^{xij}\epsilon_{j}+\tfrac{1}{\kappa^{2}}gW^{x}\epsilon^{i},$$

$$\delta q^{X}=-\mathrm{i}\bar{\epsilon}^{i}\zeta^{A}f_{iA}^{X},$$

$$\delta\zeta^{A}=\tfrac{1}{2}\mathrm{i}\gamma^{\mu}\widehat{\mathcal{D}}_{\mu}q^{X}f_{X}^{iA}\epsilon_{i}-\delta q^{X}\omega_{XB}{}^{A}\zeta^{B}+\tfrac{1}{\kappa^{2}}g\mathcal{N}_{i}^{A}\epsilon^{i}, \tag{5.8}$$

where Ω^{MN} is minus the inverse of Ω_{MN} in the sense that $\Omega_{MP}\Omega^{NP} = \delta_M{}^N$. We also denoted

$$
\begin{aligned}
\vartheta_\mu^{\widetilde{I}} &= -\tfrac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^x h_x^{\widetilde{I}} - \tfrac{\sqrt{6}}{4\kappa}\mathrm{i}h^{\widetilde{I}}\bar{\epsilon}\psi_\mu, \\
\mathcal{D}_{[\mu}\vartheta_{\nu]}^{\widetilde{I}} &= \partial_{[\mu}\vartheta_{\nu]}^{\widetilde{I}} + gA_{[\mu}^J t_{J\widetilde{K}}{}^{\widetilde{I}}\vartheta_{\nu]}^{\widetilde{K}}, \\
\widehat{\mathcal{D}}_\mu q^X &= \partial_\mu q^X + gA_\mu^I k_I^X + \mathrm{i}\bar{\psi}_\mu^j\zeta^B f_{jB}^X \\
\widehat{\mathcal{D}}_\mu\phi^x &= \partial_\mu\phi^x + gA_\mu^I K_I^x - \tfrac{1}{2}\mathrm{i}\bar{\psi}_\mu\lambda^x, \\
D_\mu(\hat{\omega})\epsilon^i &= \mathcal{D}_\mu(\hat{\omega})\epsilon^i - \partial_\mu q^X \omega_X^{ij}\epsilon_j - g\kappa^2 A_\mu^I P_I^{ij}\epsilon_j,
\end{aligned}
\tag{5.9}
$$

where $\mathcal{D}_\mu(\hat{\omega})$ is defined as in (4.2).

6 Comparison with earlier papers

In this section, we compare our results with the literature, and especially with [2] (CD) and [1,21–23,36] (GZ)[8]. To compare with these papers we put $\kappa = 1$. A notational difference with CD is that we have $\gamma^{abcde} = \mathrm{i}\varepsilon^{abcde}$, while CD uses $\gamma^{abcde} = -\mathrm{i}\varepsilon^{abcde}$. Another difference is in the symplectic metric Ω_{MN}. Our Ω_{MN} is 4 times the one in CD and in GZ. Furthermore they use $\Omega_{MN}\Omega^{NP} = \delta_M^P$ and our $t_{I\widetilde{J}}{}^{\widetilde{K}}$ is denoted as $\Lambda_{I\widetilde{J}}^{\widetilde{K}}$. The object $\tilde{B}_{ab}$ is the one denoted as B_{ab} in CD and in GZ, while $\widehat{\mathcal{H}}_{ab}^{\widetilde{I}}$ is denoted as $Q_{ab}^{\widetilde{I}}$ in CD. In CD there are two coupling constants, g and g_R, which we find to be the same.

Our results for vector and tensor multiplets without off-diagonal vector-tensor couplings ($t_{IJ}{}^M = 0$) agree with GZ. Adding the couplings to hypermultiplets we also mostly agree with CD, though some coefficients differ. For example, we differ in the coefficient in front of the hyperino mass term and in the hyperino bilinear proportional to the field strength $\mathcal{H}$.

However, the action (5.7) contains also more general couplings than previously considered because of the possibility that $t_{IJ}{}^M \neq 0$, i.e. a representation of the gauge group on the vector and tensor multiplets that is not completely reducible. For the bosonic part of the action, this is explicitly visible in the Chern-Simons couplings. Furthermore, the values of K_I^x and of W^x have implicit dependence on these representation matrices. The former appear in the covariant derivatives of the scalars $\mathcal{D}_\mu\phi^x$, thus directly influencing the fermionic supersymmetry transformations, and the latter appear in the scalar potential.

7 Simplified action for bosonic solutions

In applications where solutions to the five-dimensional $N = 2$ supergravity theory are constructed, one is mainly concerned with the bosonic terms in the action, as one often sets all fermions of the solution equal to zero. A solution to the bosonic equations of motion preserves N supercharges if there are N supersymmetry parameters ϵ^i for which the corresponding supersymmetry variation of the fermionic fields remains zero. Therefore, to search

---

[8]There is an 'alternative $N = 2$ supergravity' [37], which is very different from the theory described here.

for supersymmetric BPS solutions we only need to consider the bosonic action and the supersymmetry rules of the fermions up to terms bilinear in the fermions. For the convenience of the reader we collect these expressions in this section. This section can be a starting point for a systematic search for any kind of BPS solutions.

The bosonic part of the action reads

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\text{bos}} =\ & \tfrac{1}{2\kappa^2}R(\omega) - \tfrac{1}{4}a_{\widetilde{I}\widetilde{J}}H^{\widetilde{I}}_{\mu\nu}H^{\widetilde{J}\mu\nu} - \tfrac{1}{2}g_{xy}\mathcal{D}_\mu\phi^x\mathcal{D}^\mu\phi^y - \tfrac{1}{2}g_{XY}\mathcal{D}_\mu q^X\mathcal{D}^\mu q^Y \\
& + \frac{1}{16g}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}\widetilde{B}^M_{\mu\nu}\left(\partial_\rho\widetilde{B}^N_{\sigma\tau} + 2gt_{IJ}{}^N A^I_\rho F^J_{\sigma\tau} + gt_{IP}{}^N A^I_\rho\widetilde{B}^P_{\sigma\tau}\right) \\
& + \frac{g^2}{\kappa^4}\left(4\vec{P}\cdot\vec{P} - 2\vec{P}^x\cdot\vec{P}_x - 2W_x W^x - 2\mathcal{N}_{iA}\mathcal{N}^{iA}\right) \\
& + \tfrac{\kappa}{12}\sqrt{\tfrac{2}{3}}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A^I_\mu\left[F^J_{\nu\lambda}F^K_{\rho\sigma} + f_{FG}{}^J A^F_\nu A^G_\lambda\left(-\tfrac{1}{2}gF^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right)\right] \\
& - \tfrac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A^I_\mu A^F_\nu A^G_\lambda\left(-\tfrac{1}{2}gF^K_{\rho\sigma} + \tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right).
\end{aligned}
\tag{7.1}
$$

The covariant derivatives in this bosonic truncation are given by

$$
\mathcal{D}_\mu\phi^x = \partial_\mu\phi^x + gA^I_\mu K^x_I, \qquad \mathcal{D}_\mu q^X = \partial_\mu q^X + gA^I_\mu k^X_I.
\tag{7.2}
$$

The $N = 2$ supersymmetry rules of the fermionic fields, up to bilinears in the fermions, are given by

$$
\begin{aligned}
\delta\psi^i_\mu &= D_\mu(\omega)\epsilon^i + \tfrac{\mathrm{i}\kappa}{4\sqrt{6}}h_{\widetilde{I}}H^{\widetilde{I}\nu\rho}\left(\gamma_{\mu\nu\rho} - 4g_{\mu\nu}\gamma_\rho\right)\epsilon^i - \tfrac{1}{\kappa\sqrt{6}}\mathrm{i}gP^{ij}\gamma_\mu\epsilon_j, \\
\delta\lambda^{xi} &= -\tfrac{\mathrm{i}}{2}\not{D}\phi^x\epsilon^i + \tfrac{1}{4}\gamma\cdot H^{\widetilde{I}}h^x_{\widetilde{I}}\epsilon^i - \tfrac{1}{\kappa^2}gP^{xij}\epsilon_j + \tfrac{1}{\kappa^2}gW^x\epsilon^i, \\
\delta\zeta^A &= \tfrac{1}{2}\mathrm{i}\gamma^\mu\mathcal{D}_\mu q^X f^{iA}_X\epsilon_i + \tfrac{1}{\kappa}g\mathcal{N}^A_i\epsilon^i.
\end{aligned}
\tag{7.3}
$$

8 Conclusions

In this paper, we constructed matter-coupled $N = 2$ supergravity in five dimensions, using the superconformal approach. For the matter sector we took an arbitrary number of vector, tensor and hypermultiplets. By allowing off-diagonal vector-tensor couplings, we found more general results than currently known in the literature. Our results provide the appropriate starting point for a systematic search of BPS solutions such as domain walls or black holes. Furthermore, they can be used to study properties of the scalar potential that arises in flux-compactifications of M-theory on Calabi-Yau manifolds, or any other five-dimensional vacuum. The ingredients needed for such a search are given in the previous section.

We end with some remarks.

For simplicity, we restricted ourselves to matter couplings for which an action can be constructed. As pointed out in our previous paper [19], the superconformal approach also allows conformal matter couplings for which no such action can be defined. These theories are defined in terms of equations of motion that, without introducing more variables, cannot be integrated to an action. It would be interesting to extend the gauge-fixing procedure to

these theories[9]. In the present work we only introduced gauge-fixing conditions for theories that have an action. In fact some of the gauge-fixing choices were determined by requiring that the kinetic terms in the action had a canonical (diagonal) form. It seems reasonable that precisely the same conditions can also be applied to the theories without an action. This would lead to new Poincaré matter couplings. It is known that there are theories without an action that can be obtained via a Scherk-Schwarz reduction of a theory with an action [39–43]. It would be interesting to see whether, in the same spirit, the $D = 5$ matter couplings with no action can be obtained from some $D = 6$ matter couplings with an action.

Finally, a central role in this paper is played by the standard Weyl multiplet. We showed that a second Weyl multiplet exists, the so-called dilaton Weyl multiplet [19]. The two multiplets describe the same number of degrees of freedom but differ in the field content. *A priori* the conformal programme can also be carried out using the dilaton Weyl multiplet. It would be interesting to see whether the dilaton Weyl multiplet may lead to matter couplings which cannot be obtained by starting from the standard Weyl multiplet. In view of previous results in four dimensions for $N = 1$ [44] and $N = 2$ [35] this is not expected, but it cannot be excluded.

Acknowledgments

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[9]However, it would be natural to start also from on-shell vector multiplets, as discussed in [38].

A Notation

Curvatures. First we explain our notation for the curvature tensors. We have Riemann tensors defined on the five-dimensional spacetime and on the target space manifold spanned by the hypermultiplet scalars. As we will show below, we will use different conventions for the curvatures and Ricci tensors. They have in common, however, that compact manifolds always have a positive Ricci scalar curvature.

The 'target space notation' starts from a connection denoted by $\Gamma_{XY}{}^Z$, with Riemann curvature

$$R_{XYZ}{}^W \equiv 2\partial_{[X}\Gamma_{Y]Z}{}^W + 2\Gamma_{V[X}{}^W\Gamma_{Y]Z}{}^V. \tag{A.1}$$

The Ricci scalar and Ricci tensor on the target space are then defined as (in agreement with [19])

$$R = g^{XY}R_{XY}, \qquad R_{XY} = R_{ZXY}{}^Z. \tag{A.2}$$

Now we come to the definition of the spacetime curvature. In the general relativity literature, one usually denotes the Levi-Civita connection as $\Gamma^\rho{}_{\mu\nu}$, i.e. with the upper index on the left. The Riemann curvature is defined as

$$R^\sigma{}_{\rho\mu\nu} \equiv 2\partial_{[\mu}\Gamma^\sigma{}_{\nu]\rho} + 2\Gamma^\sigma{}_{\tau[\mu}\Gamma^\tau{}_{\nu]\rho}, \tag{A.3}$$

and has its upper index on the left, in contrast to (A.1). Spacetime Ricci tensors and scalars are then defined as

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^\rho{}_{\nu\rho\mu}. \tag{A.4}$$

Note that this is a different convention from the one used in [18], where the second and third indices are contracted to define the Ricci tensor. This means that equations (3.11) and (3.12) in [18] change sign in the above conventions. As stated before, we use these conventions such that compact manifolds have a positive Ricci scalar curvature.

$SU(2)$ **and vector indices.** At various places in the main text, we switch from SU(2) indices $i, j = 1, 2$ to vector indices with the convention

$$A_i{}^j \equiv i\vec{A} \cdot \vec{\sigma}_i{}^j, \tag{A.5}$$

where $\vec{\sigma}$ are the Pauli matrices. With these conventions, we obtain the identity

$$A_i{}^j B_j{}^k = -\vec{A} \cdot \vec{B}\delta_i{}^k - i(\vec{A} \times \vec{B}) \cdot \vec{\sigma}_i{}^k, \tag{A.6}$$

for any two vectors $\vec{A}$ and $\vec{B}$.

Lowering and raising $SU(2)$ indices is done using the ε symbol, in northwest-southeast (NW-SE) conventions,

$$A^i = \varepsilon^{ij}A_j, \qquad A_i = A^j\varepsilon_{ji}. \tag{A.7}$$

When the $SU(2)$ indices are omitted (e.g. in spinor contractions), NW-SE contractions are understood. For more details on the notation and conventions about spinors, we refer to [18].

B Conformal multiplets

In the following subsections we will repeat the relevant results for the vector-tensor multiplet and hypermultiplet from [19]. Just like in [19] we will discuss the two cases, with or without metric, separately.

B.1 Weyl multiplet

The Q- and S-supersymmetry and K-transformation rules for the independent fields of the standard Weyl multiplet are [18]

$$
\begin{aligned}
\delta e_\mu{}^a &= \tfrac{1}{2}\bar\epsilon\gamma^a\psi_\mu, \\
\delta\psi_\mu^i &= \mathcal{D}_\mu\epsilon^i + \mathrm{i}\gamma\cdot T\gamma_\mu\epsilon^i - \mathrm{i}\gamma_\mu\eta^i, \\
\delta V_\mu{}^{ij} &= -\tfrac{3}{2}\,\mathrm{i}\bar\epsilon^{(i}\phi_\mu^{j)} + 4\bar\epsilon^{(i}\gamma_\mu\chi^{j)} + \mathrm{i}\bar\epsilon^{(i}\gamma\cdot T\psi_\mu^{j)} + \tfrac{3}{2}\mathrm{i}\bar\eta^{(i}\psi_\mu^{j)}, \\
\delta T_{ab} &= \tfrac{1}{2}\mathrm{i}\bar\epsilon\gamma_{ab}\chi - \tfrac{3}{32}\,\mathrm{i}\bar\epsilon\widehat{R}_{ab}(Q), \\
\delta\chi^i &= \tfrac{1}{4}\epsilon^i D - \tfrac{1}{64}\gamma\cdot\widehat{R}^{ij}(V)\epsilon_j + \tfrac{1}{8}\mathrm{i}\gamma^{ab}\slashed{D}T_{ab}\epsilon^i - \tfrac{1}{8}\mathrm{i}\gamma^a D^b T_{ab}\epsilon^i \\
&\quad - \tfrac{1}{4}\gamma^{abcd}T_{ab}T_{cd}\epsilon^i + \tfrac{1}{6}T^2\epsilon^i + \tfrac{1}{4}\gamma\cdot T\eta^i, \\
\delta D &= \bar\epsilon\slashed{D}\chi - \tfrac{5}{3}\mathrm{i}\bar\epsilon\gamma\cdot T\chi - \mathrm{i}\bar\eta\chi, \\
\delta b_\mu &= \tfrac{1}{2}\mathrm{i}\bar\epsilon\phi_\mu - 2\bar\epsilon\gamma_\mu\chi + \tfrac{1}{2}\mathrm{i}\bar\eta\psi_\mu + 2\Lambda_{K\mu}.
\end{aligned}
\tag{B.1}
$$

The covariant derivatives are

$$
\begin{aligned}
\mathcal{D}_\mu\epsilon^i &= \partial_\mu\epsilon^i + \tfrac{1}{2}b_\mu\epsilon^i + \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon^i - V_\mu^{ij}\epsilon_j, \\
D_\mu T_{ab} &= (\partial_\mu - b_\mu)T_{ab} - 2\omega_{\mu[a}{}^c T_{b]c} - \tfrac{1}{2}\mathrm{i}\bar\psi_\mu\gamma_{ab}\chi + \tfrac{3}{32}\mathrm{i}\bar\psi_\mu\widehat{R}_{ab}(Q), \\
D_\mu\chi^i &= \left(\partial_\mu - \tfrac{3}{2}b_\mu + \tfrac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\right)\chi^i - V_\mu^{ij}\chi_j \\
&\quad - \tfrac{1}{4}\psi_\mu^i D + \tfrac{1}{64}\gamma\cdot\widehat{R}^{ij}(V)\psi_{\mu j} - \tfrac{1}{8}\mathrm{i}\gamma^{ab}\slashed{D}T_{ab}\psi_\mu^i + \tfrac{1}{8}\mathrm{i}\gamma^a D^b T_{ab}\psi_\mu^i \\
&\quad + \tfrac{1}{4}\gamma^{abcd}T_{ab}T_{cd}\psi_\mu^i - \tfrac{1}{6}T^2\psi_\mu^i - \tfrac{1}{4}\gamma\cdot T\phi_\mu^i.
\end{aligned}
\tag{B.2}
$$

The covariant curvatures $\widehat{R}(Q)$ and $\widehat{R}(V)$ are

$$
\begin{aligned}
\widehat{R}_{\mu\nu}{}^i(Q) &= R_{\mu\nu}{}^i(Q) + 2\mathrm{i}\gamma\cdot T\gamma_{[\mu}\psi_{\nu]}^i, \\
R_{\mu\nu}{}^i(Q) &= 2\partial_{[\mu}\psi_{\nu]}^i + \tfrac{1}{2}\omega_{[\mu}{}^{ab}\gamma_{ab}\psi_{\nu]}^i + b_{[\mu}\psi_{\nu]}^i - 2V_{[\mu}{}^{ij}\psi_{\nu]j} + 2\mathrm{i}\gamma_a\phi_{[\mu}^i e_{\nu]}{}^a, \\
\widehat{R}_{\mu\nu}{}^{ij}(V) &= R_{\mu\nu}{}^{ij}(V) - 8\bar\psi_{[\mu}^{(i}\gamma_{\nu]}\chi^{j)} - \mathrm{i}\bar\psi_{[\mu}^{(i}\gamma\cdot T\psi_{\nu]}^{j)}, \\
R_{\mu\nu}{}^{ij}(V) &= 2\partial_{[\mu}V_{\nu]}{}^{ij} - 2V_{[\mu}{}^{k(i}V_{\nu]k}{}^{j)} - 3\mathrm{i}\bar\phi_{[\mu}^{(i}\psi_{\nu]}^{j)}.
\end{aligned}
\tag{B.3}
$$

The expressions for the dependent fields are given in (4.1).

B.2 Vector-tensor multiplet

The supersymmetry rules for the vector-tensor multiplet coupled to the five-dimensional standard Weyl multiplet are given by

$$
\begin{aligned}
\delta A_\mu^I &= \tfrac{1}{2}\bar\epsilon\gamma_\mu\psi^I - \tfrac{1}{2}\mathrm{i}\sigma^I\bar\epsilon\psi_\mu, \\
\delta B_{ab}^M &= -\bar\epsilon\gamma_{[a}D_{b]}\psi^M - \tfrac{1}{2}\mathrm{i}\sigma^M\bar\epsilon\widehat{R}_{ab}(Q) + \mathrm{i}\bar\epsilon\gamma_{[a}\gamma\cdot T\gamma_{b]}\psi^M \\
&\quad + \mathrm{i}g\bar\epsilon\gamma_{ab}t_{(\widetilde{J}\widetilde{K})}{}^M\sigma^{\widetilde{J}}\psi^{\widetilde{K}} + \mathrm{i}\bar\eta\gamma_{ab}\psi^M, \\
\delta Y^{ij\widetilde{I}} &= -\tfrac{1}{2}\bar\epsilon^{(i}\slashed{D}\psi^{j)\widetilde{I}} + \tfrac{1}{2}\mathrm{i}\bar\epsilon^{(i}\gamma\cdot T\psi^{j)\widetilde{I}} - 4\mathrm{i}\sigma^{\widetilde{I}}\bar\epsilon^{(i}\chi^{j)} \\
&\quad - \tfrac{1}{2}\mathrm{i}g\bar\epsilon^{(i}\left(t_{[\widetilde{J}\widetilde{K}]}{}^{\widetilde{I}} - 3t_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\right)\sigma^{\widetilde{J}}\psi^{j)\widetilde{K}} + \tfrac{1}{2}\mathrm{i}\bar\eta^{(i}\psi^{j)\widetilde{I}}, \\
\delta\psi^{i\widetilde{I}} &= -\tfrac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{I}}\epsilon^i - \tfrac{1}{2}\mathrm{i}\slashed{D}\sigma^{\widetilde{I}}\epsilon^i - Y^{ij\widetilde{I}}\epsilon_j + \sigma^{\widetilde{I}}\gamma\cdot T\epsilon^i + \tfrac{1}{2}gt_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\epsilon^i + \sigma^{\widetilde{I}}\eta^i, \\
\delta\sigma^{\widetilde{I}} &= \tfrac{1}{2}\mathrm{i}\bar\epsilon\psi^{\widetilde{I}}.
\end{aligned}
\tag{B.4}
$$

The (superconformal) covariant derivatives are given by

$$
\begin{aligned}
D_\mu\sigma^{\widetilde{I}} &= \mathcal{D}_\mu\sigma^{\widetilde{I}} - \tfrac{1}{2}\mathrm{i}\bar\psi_\mu\psi^{\widetilde{I}}, \\
\mathcal{D}_\mu\sigma^{\widetilde{I}} &= (\partial_\mu - b_\mu)\sigma^{\widetilde{I}} + gt_{J\widetilde{K}}{}^{\widetilde{I}}A_\mu^J\sigma^{\widetilde{K}}, \\
D_\mu\psi^{i\widetilde{I}} &= \mathcal{D}_\mu\psi^{i\widetilde{I}} + \tfrac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{I}}\psi_\mu^i + \tfrac{1}{2}\mathrm{i}\slashed{D}\sigma^{\widetilde{I}}\psi_\mu^i + Y^{ij\widetilde{I}}\psi_{\mu j} - \sigma^{\widetilde{I}}\gamma\cdot T\psi_\mu^i \\
&\quad - \tfrac{1}{2}gt_{(\widetilde{J}\widetilde{K})}{}^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\psi_\mu^i - \sigma^{\widetilde{I}}\phi_\mu^i, \\
\mathcal{D}_\mu\psi^{i\widetilde{I}} &= \left(\partial_\mu - \tfrac{3}{2}b_\mu + \tfrac{1}{4}\gamma_{ab}\hat\omega_\mu{}^{ab}\right)\psi^{i\widetilde{I}} - V_\mu^{ij}\psi_j^{\widetilde{I}} + gt_{J\widetilde{K}}{}^{\widetilde{I}}A_\mu^J\psi^{i\widetilde{K}}.
\end{aligned}
\tag{B.5}
$$

The covariant curvature $\widehat{\mathcal{H}}_{\mu\nu}^{\widetilde{I}}$ should be understood as having components $\left(\widehat{F}_{\mu\nu}^I, B_{\mu\nu}^M\right)$ with $\widehat{F}_{\mu\nu}^I$ given by

$$
\widehat{F}_{\mu\nu}^I = 2\partial_{[\mu}A_{\nu]}^I + gf_{JK}{}^I A_\mu^J A_\nu^K - \bar\psi_{[\mu}\gamma_{\nu]}\psi^I + \tfrac{1}{2}\mathrm{i}\sigma^I\bar\psi_{[\mu}\psi_{\nu]}.
\tag{B.6}
$$

Finally, $\widehat{R}_{ab}(Q)$ is the supercovariant gravitino curvature defined in equations (3.4) and (3.18) of [18].

The vector-tensor multiplet can be realized in the absence of an action. In that case, the tensor part is realized on-shell and the corresponding equations of motion are given in [19, (4.4)].

In the presence of a fully symmetric tensor $C_{\widetilde{I}\widetilde{J}\widetilde{K}}$ the superconformal invariant action can be written down and it takes the form

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\text{Vector–Tensor}}^{\text{conf}} = \Big[&\left(-\tfrac{1}{4}\widehat{\mathcal{H}}_{\mu\nu}^{\widetilde{I}}\widehat{\mathcal{H}}^{\mu\nu\widetilde{J}} - \tfrac{1}{2}\bar\psi^{\widetilde{I}}\slashed{D}\psi^{\widetilde{J}} + \tfrac{1}{3}\sigma^{\widetilde{I}}\Box^c\sigma^{\widetilde{J}} + \tfrac{1}{6}D_a\sigma^{\widetilde{I}}D^a\sigma^{\widetilde{J}} + Y_{ij}^{\widetilde{I}}Y^{ij\widetilde{J}}\right)\sigma^{\widetilde{K}} \\
&- \tfrac{4}{3}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\left(D + \tfrac{26}{3}T_{ab}T^{ab}\right) + 4\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\widehat{\mathcal{H}}_{ab}^{\widetilde{K}}T^{ab} \\
&- \tfrac{1}{8}\mathrm{i}\bar\psi^{\widetilde{I}}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{J}}\psi^{\widetilde{K}} - \tfrac{1}{2}\mathrm{i}\bar\psi^{i\widetilde{I}}\psi^{j\widetilde{J}}Y_{ij}^{\widetilde{K}} + \mathrm{i}\sigma^{\widetilde{I}}\bar\psi^{\widetilde{J}}\gamma\cdot T\psi^{\widetilde{K}} - 8\mathrm{i}\sigma^{\widetilde{I}}\sigma^{\widetilde{J}}\bar\psi^{\widetilde{K}}\chi \\
&+ \tfrac{1}{6}\sigma^{\widetilde{I}}\bar\psi_\mu\gamma^\mu\left(\mathrm{i}\sigma^{\widetilde{J}}\slashed{D}\psi^{\widetilde{K}} + \tfrac{1}{2}\mathrm{i}(\slashed{D}\sigma^{\widetilde{J}})\psi^{\widetilde{K}} - \tfrac{1}{4}\gamma\cdot\widehat{\mathcal{H}}^{\widetilde{J}}\psi^{\widetilde{K}} + 2\sigma^{\widetilde{J}}\gamma\cdot T\psi^{\widetilde{K}} - 8\sigma^{\widetilde{J}}\sigma^{\widetilde{K}}\chi\right)
\end{aligned}
$$

$$-\frac{1}{6}\bar{\psi}_a\gamma_b\psi^{\tilde{I}}\left(\sigma^{\tilde{J}}\widehat{\mathcal{H}}^{ab\tilde{K}}-8\sigma^{\tilde{J}}\sigma^{\tilde{K}}T^{ab}\right)-\frac{1}{12}\sigma^{\tilde{I}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^{\tilde{J}}\widehat{\mathcal{H}}^{\tilde{K}}_{\mu\nu}$$

$$+\frac{1}{12}\mathrm{i}\sigma^{\tilde{I}}\bar{\psi}_a\psi_b\left(\sigma^{\tilde{J}}\widehat{\mathcal{H}}^{ab\tilde{K}}-8\sigma^{\tilde{J}}\sigma^{\tilde{K}}T^{ab}\right)+\frac{1}{48}\mathrm{i}\sigma^{\tilde{I}}\sigma^{\tilde{J}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\widehat{\mathcal{H}}^{\tilde{K}}_{\mu\nu}$$

$$-\frac{1}{2}\sigma^{\tilde{I}}\bar{\psi}^i_\mu\gamma^\mu\psi^{j\tilde{J}}Y^{\tilde{K}}_{ij}+\frac{1}{6}\mathrm{i}\sigma^{\tilde{I}}\sigma^{\tilde{J}}\bar{\psi}^i_\mu\gamma^{\mu\nu}\psi^j_\nu Y^{\tilde{K}}_{ij}-\frac{1}{24}\mathrm{i}\bar{\psi}_\mu\gamma_\nu\psi^{\tilde{I}}\bar{\psi}^{\tilde{J}}\gamma^{\mu\nu}\psi^{\tilde{K}}$$

$$+\frac{1}{12}\mathrm{i}\bar{\psi}^i_\mu\gamma^\mu\psi^{j\tilde{I}}\bar{\psi}^{\tilde{J}}_i\psi^{\tilde{K}}_j-\frac{1}{48}\sigma^{\tilde{I}}\bar{\psi}_\mu\psi_\nu\bar{\psi}^{\tilde{J}}\gamma^{\mu\nu}\psi^{\tilde{K}}+\frac{1}{24}\sigma^{\tilde{I}}\bar{\psi}^i_\mu\gamma^{\mu\nu}\psi^j_\nu\bar{\psi}^{\tilde{J}}_i\psi^{\tilde{K}}_j$$

$$-\frac{1}{12}\sigma^{\tilde{I}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^{\tilde{J}}\bar{\psi}_\mu\gamma_\nu\psi^{\tilde{K}}+\frac{1}{24}\,\mathrm{i}\sigma^{\tilde{I}}\sigma^{\tilde{J}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda}\psi^{\tilde{K}}\bar{\psi}_\mu\psi_\nu$$

$$+\frac{1}{48}\mathrm{i}\sigma^{\tilde{I}}\sigma^{\tilde{J}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar{\psi}_\mu\gamma_\nu\psi^{\tilde{K}}+\frac{1}{96}\sigma^{\tilde{I}}\sigma^{\tilde{J}}\sigma^{\tilde{K}}\bar{\psi}_\lambda\gamma^{\mu\nu\lambda\rho}\psi_\rho\bar{\psi}_\mu\psi_\nu\bigg]C_{\tilde{I}\tilde{J}\tilde{K}}$$

$$+\frac{1}{16g}e^{-1}\varepsilon^{\mu\nu\rho\sigma\tau}\Omega_{MN}\widetilde{B}^M_{\mu\nu}\left(\partial_\rho\widetilde{B}^N_{\sigma\tau}+2gt_{IJ}{}^N A^I_\rho F^J_{\sigma\tau}+gt_{IP}{}^N A^I_\rho\widetilde{B}^P_{\sigma\tau}\right)$$

$$-\frac{1}{24}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}C_{IJK}A^I_\mu\big(F^J_{\nu\lambda}F^K_{\rho\sigma}-f_{FG}{}^J A^F_\nu A^G_\lambda\big(\tfrac{1}{2}gF^K_{\rho\sigma}-\tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\big)\big)$$

$$-\frac{1}{8}e^{-1}\varepsilon^{\mu\nu\lambda\rho\sigma}\Omega_{MN}t_{IK}{}^M t_{FG}{}^N A^I_\mu A^F_\nu A^G_\lambda\left(-\tfrac{1}{2}gF^K_{\rho\sigma}+\tfrac{1}{10}g^2 f_{HL}{}^K A^H_\rho A^L_\sigma\right)$$

$$+\frac{1}{4}\mathrm{i}g\bar{\psi}^{\tilde{I}}\psi^{\tilde{J}}\sigma^{\tilde{K}}\sigma^{\tilde{L}}\left(t_{[\tilde{I}\tilde{J}]}{}^{\widetilde{M}}C_{\widetilde{M}\tilde{K}\tilde{L}}-4t_{(\tilde{I}\tilde{K})}{}^{\widetilde{M}}C_{\widetilde{M}\tilde{J}\tilde{L}}\right)$$

$$-\frac{1}{4}g\bar{\psi}_\mu\gamma^\mu\psi^{\tilde{I}}\sigma^{\tilde{J}}\sigma^{\tilde{K}}\sigma^{\tilde{L}}t_{(\tilde{J}\tilde{K})}{}^{\widetilde{M}}C_{\widetilde{M}\tilde{I}\tilde{L}}$$

$$-\frac{1}{2}g^2\sigma^I\sigma^J\sigma^K\sigma^{\widetilde{M}}\sigma^{\widetilde{N}}t_{J\widetilde{M}}{}^P t_{K\widetilde{N}}{}^Q C_{IPQ},\tag{B.7}$$

where the superconformal d'Alembertian is defined as

$$\Box^c\sigma^{\tilde{I}} = D^a D_a\sigma^{\tilde{I}}$$

$$= \left(\partial^a-2b^a+\omega_b{}^{ba}\right)D_a\sigma^{\tilde{I}}+gt_{J\tilde{K}}{}^{\tilde{I}}A^J_a D^a\sigma^{\tilde{K}}-\tfrac{1}{2}\mathrm{i}\bar{\psi}_\mu D^\mu\psi^{\tilde{I}}-2\sigma^{\tilde{I}}\bar{\psi}_\mu\gamma^\mu\chi$$

$$+\tfrac{1}{2}\bar{\psi}_\mu\gamma^\mu\gamma\cdot T\psi^{\tilde{I}}+\tfrac{1}{2}\bar{\phi}_\mu\gamma^\mu\psi^{\tilde{I}}+2f_\mu{}^\mu\sigma^{\tilde{I}}-\tfrac{1}{2}g\bar{\psi}_\mu\gamma^\mu t_{J\tilde{K}}{}^{\tilde{I}}\psi^J\sigma^{\tilde{K}},\tag{B.8}$$

and we introduced a tensor field $\widetilde{B}^M_{ab}$ as

$$B^M_{ab} = \widetilde{B}^M_{ab}-\bar{\psi}_{[a}\gamma_{b]}\psi^M+\tfrac{1}{2}\mathrm{i}\sigma^M\bar{\psi}_a\psi_b.\tag{B.9}$$

B.3   Hypermultiplet

The supersymmetry rules for the hypermultiplet coupled to the five-dimensional standard Weyl multiplet were found to be

$$\delta q^X = -\mathrm{i}\bar{\epsilon}^i\zeta^A f^X_{iA},$$

$$\widehat{\delta}\zeta^A \equiv \delta\zeta^A+\zeta^B\omega_{XB}{}^A\delta q^X$$

$$= \tfrac{1}{2}\mathrm{i}\slashed{D}q^X f^{iA}_X\epsilon_i-\tfrac{1}{3}\gamma\cdot Tk^X f^A_{iX}\epsilon^i+\tfrac{1}{2}g\sigma^I k^X_I f^A_{iX}\epsilon^i+k^X f^A_{iX}\eta^i,\tag{B.10}$$

where $\widehat{\delta}\zeta^A$ is the covariant variation of ζ^A, see section 2.3.1 of [19]. The symmetries of the system determine the (superconformal) covariant derivatives

$$
\begin{aligned}
D_\mu q^X &= \mathcal{D}_\mu q^X + \mathrm{i}\bar{\psi}^i_\mu \zeta^A f^X_{iA}, \\
\mathcal{D}_\mu q^X &= \partial_\mu q^X - b_\mu k^X - V^{jk}_\mu k^X_{jk} + g A^I_\mu k^X_I, \\
D_\mu \zeta^A &= \mathcal{D}_\mu \zeta^A - k^X f^A_{iX} \phi^i_\mu + \tfrac{1}{2}\mathrm{i}\slashed{D}q^X f^A_{iX}\psi^i_\mu + \tfrac{1}{3}\gamma\cdot T k^X f^A_{iX}\psi^i_\mu - \tfrac{1}{2}g\sigma^I k^X_I f^A_{iX}\psi^i_\mu \\
\mathcal{D}_\mu \zeta^A &= \partial_\mu \zeta^A + \partial_\mu q^X \omega_{XB}{}^A \zeta^B + \tfrac{1}{4}\hat{\omega}_\mu{}^{bc}\gamma_{bc}\zeta^A - 2b_\mu \zeta^A + g A^I_\mu t_{IB}{}^A \zeta^B.
\end{aligned}
\tag{B.11}
$$

The equations of motion for ζ^A and q^X were obtained by imposing the superconformal algebra, and are given in [19, (4.9-11)].

Note that so far we did not require the presence of an action. Introducing a metric, the locally conformal supersymmetric action is given by

$$
\begin{aligned}
e^{-1}\mathcal{L}^{\mathrm{conf}}_{\mathrm{hyper}} =&\; -\tfrac{1}{2}g_{XY}\mathcal{D}_a q^X \mathcal{D}^a q^Y + \bar{\zeta}_A \slashed{D}\zeta^A + \tfrac{4}{9}Dk^2 + \tfrac{8}{27}T^2 k^2 \\
&- \tfrac{16}{3}\mathrm{i}\bar{\zeta}_A \chi^i k^X f^A_{iX} + 2\mathrm{i}\bar{\zeta}_A \gamma\cdot T\zeta^A - \tfrac{1}{4}W_{ABCD}\bar{\zeta}^A \zeta^B \bar{\zeta}^C \zeta^D \\
&- \tfrac{2}{9}\bar{\psi}_a \gamma^a \chi k^2 + \tfrac{1}{3}\bar{\zeta}_A \gamma^a \gamma\cdot T\psi^i_a k^X f^A_{iX} + \tfrac{1}{2}\mathrm{i}\bar{\zeta}_A \gamma^a \gamma^b \psi^i_a \mathcal{D}_b q^X f^A_{iX} \\
&+ \tfrac{2}{3}f_a{}^a k^2 - \tfrac{1}{6}\mathrm{i}\bar{\psi}_a \gamma^{ab}\phi_b k^2 - \bar{\zeta}_A \gamma^a \phi^i_a k^X f^A_{iX} \\
&+ \tfrac{1}{12}\bar{\psi}^i_a \gamma^{abc}\psi^j_b \mathcal{D}_c q^Y J_Y{}^X{}_{ij}k_X - \tfrac{1}{9}\mathrm{i}\bar{\psi}^a \psi^b T_{ab}k^2 + \tfrac{1}{18}\mathrm{i}\bar{\psi}_a \gamma^{abcd}\psi_b T_{cd}k^2 \\
&- g\Big(\mathrm{i}\sigma^I t_{IB}{}^A \bar{\zeta}_A \zeta^B + 2\mathrm{i}k^X_I f^A_{iX}\bar{\zeta}_A \psi^{iI} + \tfrac{1}{2}\sigma^I k^X_I f^A_{iX}\bar{\zeta}_A \gamma^a \psi^i_a \\
&\qquad + \bar{\psi}^i_a \gamma^a \psi^{jI}P_{Iij} - \tfrac{1}{2}\mathrm{i}\bar{\psi}^i_a \gamma^{ab}\psi^j_b \sigma^I P_{Iij}\Big) \\
&+ 2gY^{ij}_I P^I_{ij} - \tfrac{1}{2}g^2 \sigma^I \sigma^J k^X_I k_{JX}.
\end{aligned}
\tag{B.12}
$$

C Identities of very special geometry

The vector multiplets are defined in terms of the symmetric real constant tensor $\mathcal{C}_{IJK}$. The independent scalars are ϕ^x, but many quantities are defined by functions $h^I(\phi)$, satisfying

$$
\mathcal{C}_{IJK}h^I(\phi)h^J(\phi)h^K(\phi) = 1,
\tag{C.1}
$$

and

$$
\begin{aligned}
h_I(\phi) &\equiv \mathcal{C}_{IJK}h^J(\phi)h^K(\phi) = a_{IJ}h^J, \qquad a_{IJ} \equiv -2\mathcal{C}_{IJK}h^K + 3h_I h_J, \\
\Gamma_{IJK} &\equiv -\sqrt{\tfrac{2}{3}}\left(\mathcal{C}_{IJK} - 9h^L \mathcal{C}_{L(IJ}h_{K)} + 9h_I h_J h_K\right), \quad R_{IJKL} = 2\Gamma_{KM[J}\Gamma_{I]L}{}^M,
\end{aligned}
\tag{C.2}
$$

where here and below I-type indices are lowered or raised with a_{IJ} or its inverse, which we assume to exist.

Define (with $,_x$ an ordinary derivative with respect to ϕ^x)

$$
h^I_x \equiv -\sqrt{\tfrac{3}{2}}h^I_{,x}(\phi),
\tag{C.3}
$$

which, due to the constraint (C.1) satisfies $h_I h_x^I = 0$, leading to

$$h_{Ix} \equiv a_{IJ} h_x^J = \sqrt{\tfrac{3}{2}} h_{I,x}(\phi). \tag{C.4}$$

We then also have

$$h^I h_{Ix} = 0, \qquad h_I h^I{}_x = 0. \tag{C.5}$$

These quantities define the metric on the scalar space, which is the pull-back of the metric a_{IJ} to the subspace defined by (C.1):

$$g_{xy} \equiv h_x^I h_y^J a_{IJ} = -2 h_x^I h_y^J \mathcal{C}_{IJK} h^K. \tag{C.6}$$

The above relations can be written in matrix form

$$\begin{pmatrix} h^I \\ h_x^I \end{pmatrix} a_{IJ} \begin{pmatrix} h^J & h_y^J \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g_{xy} \end{pmatrix}. \tag{C.7}$$

We can find the inverse of the first and third $(n+1) \times (n+1)$ matrices on the left-hand side (using $h_I^y \equiv g^{yx} h_{Ix}$)

$$\begin{pmatrix} h^I \\ h_x^I \end{pmatrix} \begin{pmatrix} h_I & h_I^y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \delta_x^y \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} h_I & h_I^x \end{pmatrix} \begin{pmatrix} h^J \\ h_x^J \end{pmatrix} = \delta_I^J. \tag{C.8}$$

Multiplying the latter equation with a_{JK} leads to

$$h_I h_J + h_I^x h_{Jx} = a_{IJ}. \tag{C.9}$$

Using the decomposition of the unity as in (C.8), we can write (with ';' a covariant derivative including a connection $h_{Jx;y} = h_{Jx,y} - \Gamma_{xy}^z h_{Jz}$ such that $g_{xy;z} = 0$)

$$h_{Ix;y} = \delta_I^J h_{Jx;y} = \left(h_I h^J + h_I^z h_z^J \right) h_{Jx;y} = \sqrt{\tfrac{2}{3}} \left(h_I h_y^J h_{Jx} + T_{xyz} h_I^z \right) = \sqrt{\tfrac{2}{3}} \left(h_I g_{xy} + T_{xyz} h_I^z \right),$$

$$h^I{}_{x;y} = -\sqrt{\tfrac{2}{3}} \left(h^I g_{xy} + T_{xyz} h^{Iz} \right),$$

$$T_{xyz} \equiv \sqrt{\tfrac{3}{2}} h_{Jx;y} h_z^J = -\sqrt{\tfrac{3}{2}} h_{Jx} h_{z;y}^J = \mathcal{C}_{IJK} h_x^I h_y^J h_z^K,$$

$$\Rightarrow \quad \Gamma_{xy}^z = h^{Iz} h_{Ix,y} - \sqrt{\tfrac{2}{3}} T_{xyw} g^{wz} = h_I^z h_{x,y}^I + \sqrt{\tfrac{2}{3}} T_{xyw} g^{wz}. \tag{C.10}$$

The tensor T_{xyz} is symmetric. Comparing (C.9) and (C.2), we obtain

$$h_I^x h_{Jx} = -2 \mathcal{C}_{IJK} h^K + 2 h_I h_J, \tag{C.11}$$

whose covariant derivative with respect to ϕ^y leads to

$$T_{xyz} h_I^x h_J^z = \mathcal{C}_{IJL} h_y^L + h_{(I} h_{J)y}. \tag{C.12}$$

Multiplying with another h_K^y, using again the two expressions for a_{IJ}, leads to

$$T_{xyz} h_I^x h_J^y h_K^z = \mathcal{C}_{IJK} + \tfrac{3}{2} a_{(IJ} h_{K)} - \tfrac{5}{2} h_I h_J h_K. \tag{C.13}$$

The curvature is

$$K_{xyzu} = R_{IJKL} h_x^I h_y^J h_z^K h_u^L = \tfrac{4}{3} \left(g_{x[u} g_{z]y} + T_{x[u}{}^w T_{z]yw} \right). \tag{C.14}$$

The domain of the variables should be limited to $h^I(\phi) \neq 0$ and the metrics a_{IJ} and g_{xy} should be positive definite. Due to relation (C.7) the latter two conditions are equivalent.

References

[1] M. Günaydin and M. Zagermann, *The gauging of five-dimensional, N = 2 Maxwell–Einstein supergravity theories coupled to tensor multiplets*, Nucl. Phys. **B572** (2000) 131–150, `hep-th/9912027`

[2] A. Ceresole and G. Dall'Agata, *General matter coupled $\mathcal{N} = 2$, $D = 5$ gauged supergravity*, Nucl. Phys. **B585** (2000) 143–170, `hep-th/0004111`

[3] L. Randall and R. Sundrum, *An alternative to compactification*, Phys. Rev. Lett. **83** (1999) 4690–4693, `hep-th/9906064`

[4] L. Randall and R. Sundrum, *A large mass hierarchy from a small extra dimension*, Phys. Rev. Lett. **83** (1999) 3370–3373, `hep-ph/9905221`

[5] A. C. Cadavid, A. Ceresole, R. D'Auria and S. Ferrara, *Eleven-dimensional supergravity compactified on Calabi-Yau threefolds*, Phys. Lett. **B357** (1995) 76–80, `hep-th/9506144`

[6] L. Jarv, T. Mohaupt and F. Saueressig, *Effective supergravity actions for flop transitions*, JHEP **12** (2003) 047, `hep-th/0310173`

[7] L. Jarv, T. Mohaupt and F. Saueressig, *M-theory cosmologies from singular Calabi-Yau compactifications*, JCAP **0402** (2004) 012, `hep-th/0310174`

[8] M. Nishimura, *Conformal supergravity from the AdS/CFT correspondence*, Nucl. Phys. **B588** (2000) 471–482, `hep-th/0004179`

[9] R. D'Auria, S. Ferrara and S. Vaulà, *Matter coupled $F(4)$ supergravity and the AdS_6/CFT_5 correspondence*, JHEP **10** (2000) 013, `hep-th/0006107`

[10] V. Balasubramanian, E. G. Gimon, D. Minic and J. Rahmfeld, *Four dimensional conformal supergravity from AdS space*, Phys. Rev. **D63** (2001) 104009, `hep-th/0007211`

[11] A. Ceresole, G. Dall'Agata, R. Kallosh and A. Van Proeyen, *Hypermultiplets, domain walls and supersymmetric attractors*, Phys. Rev. **D64** (2001) 104006, `hep-th/0104056`

[12] D. V. Alekseevsky, V. Cortés, C. Devchand and A. Van Proeyen, *Flows on quaternionic-Kähler and very special real manifolds*, Commun. Math. Phys. **238** (2003) 525–543, `hep-th/0109094`

[13] K. Behrndt and G. Dall'Agata, *Vacua of N = 2 gauged supergravity derived from non-homogeneous quaternionic spaces*, Nucl. Phys. **B627** (2002) 357–380, `hep-th/0112136`

[14] L. Anguelova and C. I. Lazaroiu, *Domain walls of N = 2 supergravity in five dimensions from hypermultiplet moduli spaces*, JHEP **09** (2002) 053, `hep-th/0208154`

[15] T. Kugo and K. Ohashi, *Off-shell $d = 5$ supergravity coupled to matter–Yang–Mills system*, Prog. Theor. Phys. **105** (2001) 323–353, `hep-ph/0010288`

[16] T. Kugo and K. Ohashi, *Supergravity tensor calculus in $5D$ from $6D$*, Prog. Theor. Phys. **104** (2000) 835–865, `hep-ph/0006231`

[17] T. Fujita and K. Ohashi, *Superconformal tensor calculus in five dimensions*, Prog. Theor. Phys. **106** (2001) 221–247, `hep-th/0104130`

[18] E. Bergshoeff, S. Cucu, M. Derix, T. de Wit, R. Halbersma and A. Van Proeyen, *Weyl multiplets of $N = 2$ conformal supergravity in five dimensions*, JHEP **06** (2001) 051, `hep-th/0104113`

[19] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, R. Halbersma, S. Vandoren and A. Van Proeyen, *Superconformal $N = 2$, $D = 5$ matter with and without actions*, JHEP **10** (2002) 045, `hep-th/0205230`

[20] E. Cremmer, *Supergravities in 5 dimensions*, in *Superspace and Supergravity*, Eds. S.W. Haking and M. Roček (Cambridge univ. press, 1981), 267.

[21] M. Günaydin, G. Sierra and P. K. Townsend, *The geometry of $N = 2$ Maxwell–Einstein supergravity and Jordan algebras*, Nucl. Phys. **B242** (1984) 244

[22] M. Günaydin, G. Sierra and P. K. Townsend, *Gauging the $d = 5$ Maxwell–Einstein supergravity theories: more on Jordan algebras*, Nucl. Phys. **B253** (1985) 573

[23] M. Günaydin and M. Zagermann, *The vacua of $5d$, $N = 2$ gauged Yang–Mills/Einstein/tensor supergravity: Abelian case*, Phys. Rev. **D62** (2000) 044028, `hep-th/0002228`

[24] B. de Wit, B. Kleijn and S. Vandoren, *Superconformal hypermultiplets*, Nucl. Phys. **B568** (2000) 475–502, `hep-th/9909228`

[25] B. de Wit, M. Roček and S. Vandoren, *Hypermultiplets, hyper-Kähler cones and quaternion-Kähler geometry*, JHEP **02** (2001) 039, `hep-th/0101161`

[26] E. Bergshoeff, S. Cucu, T. de Wit, J. Gheerardyn, S. Vandoren and A. Van Proeyen, *The map between conformal hypercomplex/hyper-Kähler and quaternionic(-Kähler) geometry*, in preparation

[27] A. Van Proeyen, *Superconformal tensor calculus in $N = 1$ and $N = 2$ supergravity*, in *Supersymmetry and Supergravity 1983*, XIXth winter school and workshop of theoretical physics Karpacz, Poland, ed. B. Milewski (World Scientific, Singapore 1983)

[28] E. Bergshoeff, E. Sezgin and A. Van Proeyen, *Superconformal tensor calculus and matter couplings in six dimensions*, Nucl. Phys. **B264** (1986) 653

[29] B. de Wit, B. Kleijn and S. Vandoren, *Rigid N = 2 superconformal hypermultiplets*, `hep-th/9808160`, in *Supersymmetries and Quantum Symmetries*, proc. Int. Sem. Dubna (1997), eds. J. Wess and E.A. Ivanov, Lecture Notes in Physics, Vol. 524 (Springer, 1999), p. 37

[30] B. de Wit, M. Roček and S. Vandoren, *Gauging isometries on hyper-Kähler cones and quaternion-Kähler manifolds*, Phys. Lett. **B511** (2001) 302–310, `hep-th/0104215`

[31] A. Swann, *HyperKähler and quaternionic Kähler geometry*, Math. Ann. **289** (1991) 421–450

[32] C. P. Boyer and K. Galicki, *3-Sasakian Manifolds*, Surveys Diff. Geom. **7** (1999) 123–184, `hep-th/9810250`

[33] R. D'Auria, S. Ferrara and P. Frè, *Special and quaternionic isometries: General couplings in N = 2 supergravity and the scalar potential*, Nucl. Phys. **B359** (1991) 705–740

[34] B. de Wit and A. Van Proeyen, *Potentials and symmetries of general gauged N = 2 supergravity – Yang-Mills models*, Nucl. Phys. **B245** (1984) 89

[35] B. de Wit, P. G. Lauwers and A. Van Proeyen, *Lagrangians of N = 2 supergravity - matter systems*, Nucl. Phys. **B255** (1985) 569

[36] M. Günaydin and M. Zagermann, *Gauging the full R-symmetry group in five-dimensional, N = 2 Yang–Mills/Einstein/tensor supergravity*, Phys. Rev. **D63** (2001) 064023, `hep-th/0004117`

[37] H. Nishino and S. Rajpoot, *Alternative N = 2 supergravity in singular five dimensions with matter/gauge couplings*, Nucl. Phys. **B612** (2001) 98–122, `hep-th/0105138`

[38] J. Gheerardyn, *A generalisation of (very) special geometry*, Nucl. Phys. **B680** (2004) 131–146, `hep-th/0310181`

[39] I. V. Lavrinenko, H. Lu and C. N. Pope, *Fibre bundles and generalised dimensional reductions*, Class. Quant. Grav. **15** (1998) 2239–2256, `hep-th/9710243`

[40] E. Bergshoeff, T. de Wit, U. Gran, R. Linares and D. Roest, *(Non-)Abelian gauged supergravities in nine dimensions*, JHEP **10** (2002) 061, `hep-th/0209205`

[41] J. Gheerardyn, *Solutions to the massive HLW IIA supergravity*, Phys. Lett. **B555** (2003) 264–270, `hep-th/0211192`

[42] C. M. Hull and A. Catal-Ozer, *Compactifications with S-duality twists*, JHEP **10** (2003) 034, `hep-th/0308133`

[43] J. Kerimo, J. T. Liu, H. Lu and C. N. Pope, *Supergravities with Minkowski × sphere vacua*, `hep-th/0402223`

[44] S. Ferrara, L. Girardello, T. Kugo and A. Van Proeyen, *Relation between different auxiliary field formulations of $N = 1$ supergravity coupled to matter*, Nucl. Phys. **B223** (1983) 191